Sums of two relatively prime k-th powers.

Roger C. Baker (Provo, UT)

§1 Introduction.

Let k be a natural number, $k \geq 3$. Let $V_k(x)$ be the number of solutions (u, v) in $\mathbb{Z}^2$ of

$$|u|^k + |v|^k \leq x, \quad (u, v) = 1$$

and let

$$E_k(x) = V_k(x) - c_k x^{2/k},$$

where $c_k = \frac{3\Gamma^2(1/k)}{\pi^2 \Gamma(2/k)}$, be the error term in the asymptotic formula for $V_k(x)$.

Recent progress in estimating $E_k(x)$ has been conditional on the Riemann hypothesis. The best currently known result for $E_3(x)$ under the Riemann hypothesis is

$$E_3(x) = O(x^{\theta_3 + \epsilon}) \tag{1.1}$$

for every $\epsilon > 0$, where $\theta_3 = 9581/36864 = 0.2599\ldots$ (Baker [2]).

Although I cannot improve (1.1) at present, I shall show that it can be proved without the full strength of the Riemann hypothesis.

Theorem 1 *Suppose that $\zeta(s)$ has no zero with real part greater than*

$$\rho_3 = \frac{123\theta_3 - 30}{90\theta_3 - 20} = 0.5802\ldots.$$

Then (1.1) *holds.*

For $E_4(x)$, we have the bound

$$E_4(x) = O(x^{\beta + \epsilon}), \ \beta = \frac{107}{512} = 0.2089\ldots \tag{1.2}$$

under the Riemann hypothesis. This result is given in Zhai [18]. (Earlier papers on $E_k(x)$ are listed in [18].) It is claimed by Zhai and Cao [20] that (1.2) holds with β replaced by $37/184 = 0.2010\ldots$, but the proof contains an error. On page 167 of [20], it is shown that

$$x^{-\epsilon} E_4(x) \ll \sum_{j=1}^{7} x^{\eta_j},$$

where the η_j are given explicitly and $\eta_4 = 37/184$. However, $\eta_7 = 0.2096\ldots$, so the result that ensues is weaker than (1.2).

In the present paper I shall obtain

$$E_4(x) = O(x^{\theta_4+\epsilon}) \tag{1.3}$$

under the Riemann hypothesis, where

$$\theta_4 = \frac{7801}{37616} = 0.2073\ldots.$$

As above, I can reach the same result with a narrower zero-free strip.

Theorem 2 *We have* (1.3) *for every* $\epsilon > 0$*, provided that* $\zeta(s)$ *has no zero with real part greater than*

$$\rho_4 = \frac{32\theta_4 - 5}{16\theta_4 - 1} = 0.7058\ldots.$$

It is of interest to examine the mean square of $E_k(x)$. The objective here is to prove a result of the form

$$\int_0^X E_k(x)^2 dx = d_k X^{1+2/k-2/k^2} + O(X^{1+2/k-2/k^2-\eta}) \tag{1.4}$$

for a positive constant η. Here

$$d_k = \frac{c_k'^2 e_k}{2\left(1 + \frac{2}{k} - \frac{2}{k^2}\right)},$$

with

$$c_k' = \frac{8\Gamma(1/k)}{\pi k}\left(\frac{k}{2\pi}\right)^{1/k}, \quad e_k = \sum_{k=1}^{\infty}\left(\sum_{d|n}\mu(d)d^{2/k}\right)^2 n^{-2-2/k}.$$

The asymptotic formula (1.4) was obtained by Zhai [19] for $k \geq 6$, and Zhai and Cao [20] for $k = 5$, under the Riemann hypothesis, with an explicitly given $\eta = \eta(k)$. In the present paper I fill in the missing cases $k = 3, 4$, and as above, assume only a narrower zero-free strip.

Theorem 3 *Suppose that $\zeta(s)$ has no zero with real part greater than χ, where $\chi < 1 - 1/k$. Then the asymptotic formula* (1.4) *holds with a positive constant $\eta = \eta(\chi, k)$.*

The proof permits the calculation of a value for $\eta(\chi, k)$. I leave some of the details of this calculation to the interested reader. The improvement over the earlier results stems from a relatively simple tool (Lemma 7 below).

Let $r_k(n)$ denote the number of representations of the positive integer n in the form

$$n = |u|^k + |v|^k, \quad (u, v) \in \mathbb{Z}^2.$$

The Dirichlet series

$$Z_k(s) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}$$

is known to have an extension to a function analytic in

$$\mathrm{Re}\, s > 1/k - 1/k^2,$$

except for a simple pole at $s = 2/k$; see, for example, Zhai [19]. To obtain our theorems, we need to study the mean value

$$M_k(\sigma, T) = \int_T^{2T} |Z_k(\sigma + it)|^2 dt.$$

I shall show that

$$M_k(\sigma, T) \ll T^{2+\epsilon}$$

for $\sigma \geq 1/k - 1/k^2 + \epsilon$. This is used in the proof of Theorem 3. The stronger estimate

$$M_k(\sigma, T) \ll T^{1+\epsilon} \tag{1.5}$$

seems inaccessible without increasing σ substantially. For Theorems 1 and 2, we need σ as small as possible in (1.5) to narrow our zero-free strip. Zhai [19] obtains (1.5) with $\sigma = \frac{3}{2k} - \frac{1}{2k^2}$.

Theorem 4 *The bound* (1.5) *holds provided that*

$$\sigma \geq 2/5 \ (k=3), \quad \sigma \geq 3/2k - 1/k^2 \ (k = 4, 5, \ldots).$$

We isolate as a theorem a result on the mean values of partial sums of $\sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}$.

Theorem 5 *Let* $\sigma \geq 2/5$ $(k=3)$, $\sigma \geq (4k-4)/k(3k-2)$ $(k \geq 4)$. *Let*

$$\alpha = \max\left(\frac{4}{k} - 2\sigma, \ 3 - 2\sigma k\right).$$

Suppose that $X \geq 1$ *and* $X^{\alpha} \leq T$. *Then*

$$\int_T^{2T} \left| \sum_{n \leq X} \frac{r_k(n)}{n^{\sigma+it}} \right|^2 dt \ll T^{1+\epsilon}.$$

This result is used in the proof of Theorem 4.

Most of the estimates for exponential sums and integrals used below can be traced back to the ideas of van der Corput. However, the paper of Robert and Sargos [12] not only plays an important role in a result from [2] re-used here, but is used afresh. In particular, an exponential sum estimate based on counting solutions of

$$\left| \frac{(h_1^q + \ell_1^q)^{1/q}}{n_1} - \frac{(h_2^q + \ell_2^q)^{1/q}}{n_2} \right| < \Delta,$$

in the proof of Theorem 1, depends on [12].

Constants implicit in the 'O' and '$\ll$' notations may depend (unless otherwise stated) on k and ϵ; other dependencies are made explicit where they occur. Let $C(k)$ be a sufficiently large positive constant depending on k. We write $A \asymp B$ for $A \ll B \ll A$. The notation '$n - a \sim N$' (where n is an integer variable and a is fixed) means $N < n - a \leq 2N$. We write $e(z)$ for $e^{2\pi i z}$.

I would like to acknowledge the friendly hospitality of the Department of Mathematics, University of Florida, where much of the work was accomplished.

§2 Preliminary results

Let us write $\psi(w) = w - [w] - 1/2$, with $[\dots]$ the integer part function. The nice paper of Kühleitner [11] is a helpful source for the present topic. We find there the formula

$$T_k(x) = A_k x^{2/k} + c_k' \Phi_k(x^{2/k}) x^{1/k - 1/k^2} + P_k(x^{2/k}) + B_k(x) \tag{2.1}$$

for the summatory function $T_k(x) = \sum_{n \le x} r_k(n)$. Here $A_k = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}$,

$$\Phi_k(u) = \sum_{m=1}^{\infty} m^{-1-1/k} \cos 2\pi \left(m u^{1/2} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right),$$

$$P_k(u) = -8 \sum_{2^{-1/k} u^{1/2} \le n \le u^{1/2}} \psi((u^{k/2} - n^k)^{1/k}), \tag{2.2}$$

and $B_k(x) = O(1)$.

Kuba [10] has shown that

$$P_k(u) = O(u^{23/73+\epsilon}). \tag{2.3}$$

Presumably this could be sharpened by a careful application of the recent work of Huxley [7] within the argument of [10]. Kühleitner [11] gives an asymptotic formula for the mean value of $P_k(u)$,

$$\int_0^X P_k(u)^2 du = C_k X^{3/2} + O(X^{3/2-\delta_k}) \tag{2.4}$$

where C_k and δ_k are positive numbers given explicitly.

For u in a range $[U, 2U]$, U large and positive, Kühleitner splits up the interval of summation in (2.2) using subintervals $[N_r, N_{r+1}]$, where

$$N_r = N_r(u) = \frac{u^{1/2}}{(1 + 2^{-rq})^{1/k}}, r = 0, 1, \dots, R.$$

Here $q = k/(k-1)$ and R is the least integer such that

$$\sqrt{u} - N_R < 1 \quad \text{for } u \in [U, 2U].$$

It is easy to see that

$$N_{r+1} - N_r = O(U^{1/2} 2^{-rq}),$$

$$2^{qR} \asymp U^{1/2} \quad , \; R = O(\log U), \tag{2.5}$$

and that

$$P_k(u) = -8 \sum_{r=0}^{R} \sum_{n=N_r}^{N_{r+1}} \psi\left((u^{k/2} - n^k)^{1/k}\right) + O(\log U). \tag{2.6}$$

There are two well-known approximations to ψ. The first is elementary (Jones [9]):

$$\int_0^1 \left| \psi(w) + \sum_{0<|h|\le H} \frac{e(hw)}{h} \right|^2 dw \ll H^{-1}. \tag{2.7}$$

The second, due to Vaaler [15], is likewise important in the present paper:

$$\left| \psi(w) - \sum_{0<|h|\le H} a_h e(hw) \right| \le B(w), \tag{2.8}$$

where $B(w) = \sum_{|h|\le H} b_h e(hw)$ is a non-negative trigonometric polynomial, and

$$a_h \ll \frac{1}{h} \; , \; b_h \ll \frac{1}{H}. \tag{2.9}$$

(The a_h and b_h are given explicitly by Vaaler. See also the appendix to [3].) It is worth noting that (2.8), (2.9) are valid even when $H < 1$, since $|\psi(w)| \le 1/2$.

Thus for $U \le u \le 2U$, and $H_r \ge 1$ $(0 \le r \le R)$

$$\left| P_k(u) + 8 \sum_{r=0}^{R} \sum_{0<|h|\le H_r} a_h \sum_{n=N_r}^{N_{r+1}} e\left(h(u^{k/2} - n^k)^{1/k}\right) \right| \tag{2.10}$$

$$\le \sum_{r=0}^{R} \sum_{|h|\le H_r} b_h \sum_{n=N_r}^{N_{r+1}} e\left(h(u^{k/2} - n^k)^{1/k}\right) + C(k) \log U.$$

Moreover, the van der Corput B-process yields

$$\sum_{n=N_r}^{N_{r+1}} e(h(u^{k/2} - n^k)^{1/k}) = \frac{e(-1/8)}{\sqrt{k-1}}\, h\, u^{1/4} \sideset{}{''}\sum_{m\in[h2^r, h2^{r+1}]} (hm)^{-1+q/2} \times \tag{2.11}$$

$$\times\, |(h,m)|^{-q+1/2} e(-u^{1/2}\, |(h,m)|) + O(\log(|h|U + 2)).$$

Here and subsequently,

$$|(h,m)| = (|h|^q + |m|^q)^{1/q},$$

and $\sum''$ indicates that the first and last terms are weighted with a factor 1/2. See Kühleitner [11] for more details.

It is convenient to write $\Delta_k(x) = T_k(x) - A_k x^{2/k}$.

For $y > 1$, let $f(y,s)$ denote the meromorphic function

$$f(y,s) = \frac{1}{\zeta(s)} - \sum_{n\le y} \mu(n)^{-s}.$$

Lemma 1 *Let $X \ge 1$. The function $Z_k(s)$ has a meromorphic continuation to the region*

$$\operatorname{Re} s > \frac{1}{k} - \frac{1}{k^2}$$

given by

$$Z_k(s) = \sum_{n\le X} \frac{r_k(n)}{n^s} + \frac{2}{k}\,\frac{A_k X^{2/k-s}}{s-2/k} - X^{-s}\Delta_k(X) + s\int_X^\infty \frac{\Delta_k(\omega)}{\omega^{s+1}}\,d\omega.$$

Proof. See, for example, the proof of Lemma 3.1 of Zhai [19].

Lemma 2 *Let $y > 1$. For a suitable positive constant $C = C(k)$, we have*

$$E_k(x) = \sum_{d\le y} \mu(d)\Delta_k\left(\frac{x}{d^k}\right) + \frac{1}{2\pi i}\int_{\lambda-ix^C}^{\lambda+ix^C} f(y,ks)Z_k(s)\,\frac{x^s}{s}\,ds + O(1)$$

whenever $\frac{1}{k} - \frac{1}{k^2} + \epsilon \le \lambda \le \frac{2}{k} - \epsilon$.

Proof. This can easily be adapted from the proof of Lemma 19 of [2], for example.

Lemma 3 *Let $\epsilon > 0$. Suppose that $\zeta(s)$ has no zero with $\operatorname{Re} s > \theta$, where $\frac{1}{2} \le \theta < 1-\epsilon$. Then $\zeta(s)$ and $\zeta(s)^{-1}$ are $O(t^\epsilon)$ for $s = \sigma + it$, $t \ge 2$, $\sigma \ge \theta + \epsilon$.*

Proof. In view of results in Titchmarsh [14], Chapter 5, we suppose that $\sigma < 1$. Following Titchmarsh [14], §14.2, we apply the Borel-Carathéodory theorem ([13], §5.5) to the function $\log \zeta(z)$ and the circles with center $2 + it$

and radii $2-\theta-\frac{\delta}{2}$, $2-\theta-\delta$, where $0<\delta<1-\epsilon$. On the larger circle, writing $B_1, B_2, \ldots$ for absolute constants,

$$\mathrm{Re}(\log\zeta(z)) = \log\,|\zeta(z)| < B_1 \log t.$$

Hence, on the smaller circle,

$$\begin{aligned} |\log\zeta(z)| &\le \frac{4-2\theta-2\delta}{\delta/2} B_1 \log t + \frac{4-2\theta-3\delta/2}{\delta/2}\,|\log\zeta(2+it)| \\ &< B_2\delta^{-1}\log t. \end{aligned}$$

In particular, we find that

$$|\log\zeta(\sigma+it)| < B_2\delta^{-1}\log t. \tag{2.12}$$

Now let $\sigma_1 = \epsilon^{-3}$ and $\delta = \epsilon^4$, and apply Hadamard's three-circles theorem ([13], §5.3) to circles of center $\sigma_1 + it$ and radii $r_1 < r_2 < r_3$,

$$r_1 = \sigma_1 - 1 - \delta,\ r_2 = \sigma_1 - \sigma,\ r_3 = \sigma_1 - \theta - \delta.$$

If the maxima of $|\log\zeta(z)|$ on the respective circles are M_1, M_2, M_3, we obtain

$$M_2 \le M_1^{1-a} M_3^a,$$

where

$$\begin{aligned} a = \frac{\log r_2/r_1}{\log r_3/r_1} &= \frac{\log\left(1+\frac{1+\delta-\sigma}{\sigma_1-1-\delta}\right)}{\log\left(1+\frac{1-\theta}{\sigma_1-1-\delta}\right)} \\ &= \frac{1-\sigma+\delta}{1-\theta} + O(\sigma_1^{-1}) \\ &= \frac{1-\sigma+O(\epsilon^2)}{1-\theta}. \end{aligned}$$

The last two implied constants are absolute.

By (2.12), $M_3 < B_2\delta^{-1}\log t$, and it is easy to show (see [14], §14.2) that $M_1 < B_3\delta^{-1}$. Since $\sigma+it$ is on the middle circle,

$$\begin{aligned} |\log\zeta(\sigma+it)| &< \left(\frac{B_3}{\delta}\right)^{1-a}\left(\frac{B_2\log t}{\delta}\right)^a \\ &< C(\epsilon)(\log t)^{(1-\sigma+\epsilon/2)/(1-\theta)}. \end{aligned}$$

This is stronger than the required bound.

Lemma 4 *Suppose that $\zeta(s)$ has no zero with $\operatorname{Re} s > \theta$, where $\frac{1}{2} \le \theta < 1-\epsilon$. Then*

$$f(y,s) = O(y^{\theta-\sigma+\epsilon}|t|^{\epsilon}) \tag{2.13}$$

for $y > 1$, $s = \sigma + it$, $\theta + \epsilon \le \sigma \le k$, $|t| \ge 2$.

Proof. It suffices to prove (2.13) when y is half an odd integer. In Lemma 3.12 of [14], take $a_n = \mu(n)$, $f(s) = \frac{1}{\zeta(s)}$, $c = 2$. We obtain

$$\sum_{n<y} \frac{\mu(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \, \frac{x^w}{w} \, dw + O\left(\frac{y^2}{T}\right)$$

for $T > 0$. Take $T = y^{k+2}$, so that

$$\frac{y^2}{T} = O(y^{\theta-\sigma}).$$

We have

$$\begin{aligned} &\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \, \frac{y^w}{w} \, dw \\ &= \left(\int_{\theta+\frac{\epsilon}{2}-\sigma-iT}^{\theta+\frac{\epsilon}{2}-\sigma+iT} - \int_{\theta+\frac{\epsilon}{2}-\sigma-iT}^{2-iT} + \int_{\theta+\frac{\epsilon}{2}-\sigma+iT}^{2+iT} \right) \frac{1}{\zeta(s+w)} \, \frac{y^w}{w} \, dw. \end{aligned} \tag{2.14}$$

We may now apply Lemma 3. The horizontal integrals on the right-hand side of (2.14) are

$$\begin{aligned} O\left(T^{-1+\epsilon} \int_{\theta+\frac{\epsilon}{2}-\sigma}^{2} y^u du \right) &= O(T^{-1+\epsilon} y^2) \\ &= O(y^{\theta-\sigma}). \end{aligned}$$

The vertical integral is

$$O\left(y^{\theta+\frac{\epsilon}{2}-\sigma} \int_{-T}^{T} (1+|t|)^{-1+\epsilon/4k} dt \right) = O(y^{\theta-\sigma+\epsilon}).$$

The lemma follows on combining these estimates.

Lemma 5 *Let $A > 0$, $A < B \le 2A$, $C \ge 2$, $C < D \le 2C$. Let f be a bounded measurable function on $[A, B]$. Then*

$$\int_C^D \left| \int_A^B f(x) x^{it} dx \right|^2 dt \ll A \log C \int_A^B |f(x)|^2 dx.$$

Proof. We have

$$\begin{aligned}
&\int_C^D \left| \int_A^B f(x) x^{it} dx \right|^2 dt \\
&\quad = \int_A^B \int_A^B f(x_1) f(x_2) \int_C^D \left(\frac{x_1}{x_2} \right)^{it} dt\, dx_1\, dx_2
\end{aligned}$$

(by Fubini's theorem)

$$\begin{aligned}
&\le \int_A^B \int_A^B (|f(x_1)|^2 + |f(x_2)|^2) \min\left(C, \frac{1}{|\log x_1/x_2|} \right) dx_1\, dx_2 \\
&= 2 \int_A^B |f(x_1)|^2 \int_A^B \min\left(C, \frac{1}{|\log x_1/x_2|} \right) dx_2\, dx_1.
\end{aligned}$$

In the inner integral, substitute $v = x_1/x_2$; then $|\log v| \asymp |v - 1|$, so that

$$\begin{aligned}
\int_A^B \min\left(C, \frac{1}{|\log x_1/x_2|} \right) dx_2 &\ll A \int_{\frac{1}{2}}^2 \min\left(C, \frac{1}{|v-1|} \right) dv \\
&\ll A \log C.
\end{aligned}$$

The lemma follows at once.

Lemma 6 *For $1/k - 1/k^2 + \epsilon \le \sigma \le 1$ and $T \ge 2$,*

$$M_k(\sigma, T) \ll T^2 \log T.$$

Proof. By Lemma 1 with $X = 1$, $s = \sigma + it$, $T \le t \le 2T$,

$$Z_k(s) = s \int_1^\infty \frac{\Delta_k(\omega)}{\omega^{\sigma+it+1}} d\omega + O(1).$$

Hence Cauchy's inequality yields

$$M_k(\sigma, T) \ll T + T^2 \int_T^{2T} \left| \sum_{j=1}^{\infty} \int_{2^{j-1}}^{2^j} \frac{\Delta_k(\omega)_{d\omega}}{\omega^{\sigma+it+1}} \right|^2 dt$$

$$\ll T + T^2 \int_T^{2T} \left(\sum_{j=1}^{\infty} j^{-2} \right) \left(\sum_{j=1}^{\infty} j^2 \left| \int_{2^{j-1}}^{2^j} \frac{\Delta_k(\omega)_{d\omega}}{\omega^{\sigma+it+1}} \right|^2 \right) dt.$$

Since

$$\Delta_k(\omega) \ll \omega^{1/k-1/k^2}$$

from (2.1), (2.3), we find that

$$\int_{2^{j-1}}^{2^j} \frac{|\Delta_k(\omega)|^2}{\omega^{2\sigma+2}} \, d\omega \ll 2^{-j(1+\epsilon)}.$$

Applying Lemma 5,

$$\int_T^{2T} \left| \int_{2^{j-1}}^{2^j} \frac{\Delta_k(\omega)}{\omega^{\sigma+it+1}} \right|^2 dt \ll 2^{-j\epsilon} \log T$$

and

$$M_k(\sigma, T) \ll T + T^2 \left(\sum_{j=1}^{\infty} j^2 2^{-j\epsilon} \right) \log T \ll T^2 \log T.$$

Lemma 7 *Let $D > C \geq 2$, $B > A > 1$ and suppose that $g(t)$ is a bounded measurable function on $[C, D]$. Then*

$$\int_A^B \left| \int_C^D g(t) x^{it} dt \right|^2 dx \ll B \log D \int_C^D |g(t)|^2 dt.$$

Proof. This is a slight variant of Harman [4], Lemma 9.1.

Lemma 8 *Let F, G be real differentiable functions on $[a, b]$ such that G/F' is monotonic and either $F'/G \geq M > 0$, or $F'/G \leq -M < 0$. Then*

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq \frac{4}{M}.$$

Proof. This is Lemma 4.3 of [14].

Lemma 9 *Let F be a real differentiable function in $[a,b]$, such that F' is monotonic and $0 < M \leq |F'| \leq 1-\epsilon$. Then*

$$\sum_{a<n\leq b} e(f(n)) = O(M^{-1}).$$

Proof. This result is known as the Kusmin-Landau theorem. It is a consequence of Lemma 8 in conjunction with Lemma 4.8 of [14]; there is a different proof in [3].

For $H \geq 1$, $K \geq 1$, $P \geq 1$, $Q \geq 1$ and a given quadruple of real numbers $\mathbf{a} = (a_1, a_2, a_3, a_4)$, let us write

$$\mathcal{N}(\mathbf{a}, H, K, P, Q, \Delta)$$

for the number of quadruples (m_1, m_2, m_3, m_4) with $m_1 \sim H$, $m_2 \sim PH$, $m_3 \sim K$, $m_4 \sim QK$,

$$|a_1 m_1^q + a_2 m_2^q + a_3 m_3^q + a_4 m_4^q| \leq \Delta (PH)^q.$$

We write more succinctly

$$\mathcal{N}(N, \Delta) = \mathcal{N}((1, 1, -1-1), N, N, 1, 1, \Delta).$$

Lemma 10 *Suppose that $0 < c_1 \leq |a_j| \leq c_2$ $(j = 1, \ldots, 4)$ and $c_1 PH \leq QK \leq c_2 PH$. Then*

$$\mathcal{N}(\mathbf{a}, H, K, P, Q, \Delta) \ll (PH)^\epsilon (PH^2 + \Delta P^3 H^4)^{1/2} (QK^2 + \Delta Q^3 K^4)^{1/2}.$$

Here and in the proof, the implied constants depend on c_1 and c_2.

Proof. Let $M_1 = H$, $M_2 = PH$, $M_3 = K$, $M_4 = QK$, and

$$S_j(u) = \sum_{m \sim M_j} e(um^q).$$

By a slight variant of Lemma 2.1 of [16],

$$\mathcal{N}(\mathbf{a}, H, K, P, Q, \Delta) \leq \pi^2 \Delta (PH)^q \int_0^{1/2\Delta(PH)^q} \prod_{j=1}^4 |S_j(a_j u)|\, du.$$

By Hölder's inequality

$$\int_0^{\Delta(PH)^q/2} \prod_{j=1}^4 |S_j(a_j u)|\, du \le \prod_{j=1}^4 \left(\int_0^{1/2\Delta(PH)^q} |S_j(a_j u)|^4 du \right)^{1/4}$$

$$\ll \prod_{j=1}^4 \left(\int_0^{a_j/2\Delta(PH)^q} |S_j(u)|^4 du \right)^{1/4},$$

so that

$$\mathcal{N}(\mathbf{a}, H, K, P, Q, \Delta) \ll \prod_{j=1}^4 \left(\Delta(PH)^q \int_0^{c_2/2\Delta(PH)^q} |S_j(u)|^4 du \right)^{1/4}. \tag{2.15}$$

Again by Lemma 2.1 of [16],

$$\Delta(PH)^q \int_0^{c_2/2\Delta(PH)^q} |S_j(u)|^4\ du \ll \mathcal{N}\left(M_j,\ \frac{\Delta(PH)^q}{c_2 M_j^q} \right). \tag{2.16}$$

We now apply the inequality

$$\mathcal{N}(M_j, \eta_j) \ll M_j^{2+\epsilon} + M_j^{4+\epsilon}\eta_j,$$

which is Theorem 2 of Robert and Sargos [12]. We take

$$\eta_j M_j^q = \frac{4\Delta(PH)^q}{c_2}.$$

Thus, since $PH \asymp QK$,

$$\mathcal{N}\left(M_1, \frac{\Delta(PH)^q}{c_2 M_1^q} \right) \ll H^{2+\epsilon} + H^{4+\epsilon}(P^q\Delta),$$

$$\mathcal{N}\left(M_2, \frac{\Delta(PH)^q}{c_2 M_2^q} \right) \ll (PH)^{2+\epsilon} + (PH)^{4+\epsilon}\Delta,$$

$$\mathcal{N}\left(M_3, \frac{\Delta(PH)^q}{c_2 M_3^q} \right) \ll K^{2+\epsilon} + K^{4+\epsilon}(Q^q\Delta)$$

and

$$\mathcal{N}\left(M_4, \frac{\Delta(PH)^q}{c_2 M_4^q}\right) \ll (QK)^{2+\epsilon} + (QK)^{4+\epsilon}\Delta.$$

Moreover,

$$\begin{aligned}
&((PH)^2 + (PH)^4\Delta)^{1/4}(H^2 + H^4P^q\Delta)^{\frac{1}{4}} \\
&\ll (P^2H^4 + P^4H^6\Delta + P^{4+q}H^8\Delta^2)^{\frac{1}{4}} \\
&\ll (P^2H^4 + P^6H^8\Delta^2)^{1/4} \ll (PH^2 + P^3H^4\Delta)^{1/2}.
\end{aligned}$$

There is a similar bound

$$\begin{aligned}
((QK)^2 + (QK)^4\Delta)^{1/4}&(K^2 + K^4Q^q\Delta)^{1/4} \\
&\ll (QK^2 + Q^3K^4\Delta)^{1/2},
\end{aligned}$$

so that (2.16) gives the desired bound for the right-hand side of (2.15).

Lemma 11 *Let* $1 \le H \le PH$, $N \ge 1$. *The number of solutions* $\mathcal{N}$ *of*

$$\left|\frac{|(h_1, \ell_1)|}{n_1} - \frac{|(h_2, \ell_2)|}{n_2}\right| < \frac{\Delta PH}{N} \tag{2.17}$$

with $H \le h_i \le 2H$, $PH \le \ell_i \le 2PH$, $N \le n_i < 2N$ *is*

$$O((PH)^\epsilon(P^3H^4N^2\Delta + P^{3/2}H^3N)).$$

Proof. Let d be an integer in $[1, 2N)$. We count the number of solutions $\mathcal{N}_d$ of (2.17) with $(n_1, n_2) = d$. Write $n_j = k_j d$, $(k_1, k_2) = 1$, $k_1 \le 2N/d$, $k_2 \le 2N/d$.

First fix k_1, k_2. Then (2.17) implies

$$|(h_1, \ell_1)| - \frac{k_1}{k_2}\,|(h_2, \ell_2)| \ll \Delta PH \tag{2.18}$$

and indeed

$$h_1^q + \ell_1^q - \left(\frac{k_1}{k_2}\right)^q (h_2^q + \ell_2^q) \ll \Delta(PH)^q. \tag{2.19}$$

By Lemma 10 the number of solutions h_1, h_2, ℓ_1, ℓ_2 of (2.19) is

$$\ll (PH)^{\epsilon/2}(PH^2 + \Delta P^3 H^4).$$

Hence

$$\mathcal{N}_d \ll (PH)^{\epsilon/2}\left(\frac{N^2PH^2}{d^2} + \frac{\Delta N^2 P^3 H^4}{d^2}\right).$$

On the other hand, if we fix h_1, ℓ_1, h_2, ℓ_2, then (2.18) implies

$$\left|\frac{|(h_1,\ell_1)|}{|(h_2,\ell_2)|} - \frac{k_1}{k_2}\right| \leq 2\Delta.$$

Since the numbers k_1/k_2 are spaced at least $d^2/4N^2$ apart, the number of solutions of the last inequality is

$$\ll \frac{\Delta N^2}{d^2} + 1.$$

Hence

$$\mathcal{N}_d \ll P^2H^4\left(\frac{\Delta N^2}{d^2} + 1\right)$$

and indeed

$$\begin{aligned}\mathcal{N}_d &\ll (PH)^{\epsilon}\left(\frac{\Delta N^2P^3H^4}{d^2} + \min\left(\frac{N^2PH^2}{d^2}, P^2H^4\right)\right)\\ &\ll (PH)^{\epsilon}\left(\frac{\Delta N^2P^3H^4}{d^2} + \left(\frac{N^2PH^2}{d^2}\right)^{1/2}(P^2H^4)^{1/2}\right).\end{aligned}$$

The lemma follows on summing this bound over d.

Lemma 12 *Let f be a complex-valued function on $[D, D')$, where $2 \leq D < D' \leq 2D$. Suppose that $0 < U \leq D^{1/3}$, $B > 0$, and*

$$\sum_{m\sim M} a_m \sum_{\substack{n\sim N\\ D\leq mn<D'}} f(mn) \ll B$$

whenever $MN \asymp D$, $N \gg DU^{-1}$ and $|a_m| \leq 1$. Suppose further that

$$\sum_{m\sim M} a_m \sum_{\substack{n\sim N\\ D\leq mn<D'}} b_n\, f(mn) \ll B$$

whenever $MN \asymp D$, $U \ll N \ll D^{1/2}$ *and* $|a_m| \le 1$, $|b_n| \le 1$. *Then*

$$\sum_{D \le d < D'} \mu(d) f(d) \ll B D^{\epsilon}.$$

Proof. This is essentially Lemma 2(ii) of [2]. (The idea is much older; see [4] for a broader discussion.)

Lemma 13 *Let* (κ, λ) *be an exponent pair. Let* α, β *be constants,* $\alpha \neq 0$, $\alpha < 1$, $\beta < 0$. *Let* $X > 0$, $M \ge 1/2$, $N \ge 1/2$, $MN \asymp D$, $N_0 = \min(M, N)$, $L = \log(D+2)$. *Let* $|a_m| \le 1$, $|b_n| \le 1$, $I_m \subseteq (N, 2N]$, *and*

$$S_1 = \sum_{m \sim M} a_m \sum_{n \in I_m} e\left(\frac{X m^{\beta} n^{\alpha}}{M^{\beta} N^{\alpha}}\right),$$

$$S_2 = \sum_{\substack{m \sim M \\ D < mn \le D'}} a_m \sum_{n \sim N} b_n e\left(\frac{X m^{\beta} n^{\alpha}}{M^{\beta} N^{\alpha}}\right).$$

(i) We have

$$S_1 \ll L^2 \{ D N^{-1/2} + D X^{-1} + (D^{4+4\kappa} X^{1+2\kappa} N^{-(1+2\kappa)} N_0^{2(\lambda-\kappa)})^{1/(6+4\kappa)} \}.$$

(ii) If $N \ll M$ *and* $X \gg D$, *we have*

$$S_2 \ll L^{7/4} (D N^{-1/2} + D M^{-1/4} + (D^{11+10\kappa} X^{1+2\kappa} N^{2(\lambda-\kappa)})^{1/(14+12\kappa)}).$$

The implied constants depend on α, β, κ *and* λ.

Proof. See [2], Theorems 4 and 5.

Lemma 14 *Let* α, β *be real constants with* $\alpha\beta(\alpha-1)(\beta-1) \neq 0$. *Let* κ, λ, X, M, N, L, S_2 *be as in Lemma 13. Then*

$$\begin{aligned} S_2 \ll L^3 \{ & (X^{2+4\kappa} M^{8+10\kappa} N^{9+11\kappa+\lambda})^{1/(12+16\kappa)} + X^{1/6} M^{2/3} N^{3/4+\lambda/(12+12\kappa)} \\ & + (X M^3 N^4)^{1/5} + (X M^7 N^{10})^{1/11} + M^{2/3} N^{11/12+\lambda/(12+12\kappa)} \\ & + M N^{1/2} + (X^{-1} M^{14} N^{23})^{1/22} + X^{-1/2} M N \}. \end{aligned}$$

Proof. At the cost of a factor L, we can remove the condition $D < mn \le D'$ from the sum S_2. See [4], pp. 49–50. Now the result follows at once from Theorem 2 of [17].

We recall some facts about Riemann-Stieltjes integrals $\int_a^b f(t)d\alpha(t)$, as presented in Apostol [1], Chapter 9. Sometimes these integrals do not receive enough care in the number theory literature. The functions f and α are assumed to be real-valued and bounded on $[a,b]$. We must be careful to avoid both α and f being discontinuous from (e.g.) the left at any point, since then $\int_a^b f(t)d\,\alpha(t)$ may not exist (see [1], Theorem 9.28). If we begin with f a function of bounded variation continuous from the left, and α the sum of continuous function and a step function continuous from the right, then $I = \int_a^b f(t)d\,\alpha(t)$ does exist. Moreover, $J = \int_a^b \alpha(t)d\,f(t)$ exists and

$$I + J = f(b)\alpha(b) - f(a)\alpha(a)$$

([1], Theorems 9.2, 9.6, 9.11 and 9.21). Moreover, if it happens that f is continuously differentiable on $[a,b]$, then

$$\int_a^b \alpha(t)\,d\,f(t) = \int_a^b \alpha(t)f'(t)\,dt$$

([1], Theorem 9.8). We now derive some basic inequalities for the Riemann-Stieltjes integrals $\int_X^{2X} f(t)d\Delta_k(t)$ that we shall encounter. Here $X \ge 1$. From the definition, $\Delta_k(t) = T_k(t) - A_k t^{2/k}$ is the sum of a continuous function and a step function continuous from the right.

Lemma 15 *Let $f(t)$, $g(t)$ be real functions of bounded variation continuous from the left on $[X,2X]$, $|f(t)| \le g(t)$ $(t \in [X,2X])$. Then*

(i) We have

$$\int_X^{2X} f(t)\ d\Delta_k(t) \ll \|f\|_\infty X^{2/k}.$$

(ii) We have

$$\int_X^{2X} f(t)\,d\Delta_k(t) \ll \int_X^{2X} g(t)\,t^{2/k-1}dt + \left|\int_X^{2X} g(t)\,d\Delta_k(t)\right|.$$

(iii) If f is continuously differentiable on $[X,2X]$, then

$$\int_X^{2X} f(t)\,d\Delta_k(t) \ll \|f\|_\infty X^{1/k-1/k^2} + \left|\int_X^{2X} f'(t)\,\Delta_k(t)dt\right|.$$

Here $\|f\|_\infty = \sup_{X\le t\le 2X} |f(t)|$.

Proof. (i) We have

$$\int_X^{2X} f(t)d\Delta_k(t) = -\int_X^{2X} f(t)d(A_k t^{2/k}) + \int_X^{2X} f(t)dT_k(t). \tag{2.20}$$

and ([1], Theorem 9.23)

$$\left|\int_X^{2X} f(t)d(A_k t^{2/k})\right| \ll \|f\|_\infty A_k((2X)^{2/k} - X^{2/k}),$$
$$\left|\int_X^{2X} f(t)d(T_k(t))\right| \le \|f\|_\infty (T_k(2X) - T_k(X)).$$

Now (i) follows from simple bounds for the expressions used to bound the two integrals.

(ii) From (2.20), and [1], Theorem 9.22,

$$\begin{aligned}\left|\int_X^{2X} f(t)d\Delta_k(t)\right| &\le \frac{2A_k}{k}\left|\int_X^{2X} f(t)t^{2/k-1}dt\right| + \int_X^{2X} g(t)d\,T_k(t)\\ &\le \frac{2A_k}{k}\int_X^{2X} g(t)t^{2/k-1}dt + \int_X^{2X} g(t)d\,T_k(t)\\ &= \frac{4A_k}{k}\int_X^{2X} g(t)t^{2/k-1}dt + \int_X^{2X} g(t)d\Delta_k(t).\end{aligned}$$

(iii) We have

$$\int_X^{2X} f(t)d\Delta_k(t) = \Delta_k(t)f(t)\Big|_X^{2X} - \int_X^{2X}\Delta_k(t)f'(t)dt$$

$$\begin{aligned}\left|\int_X^{2X} f(t)d\Delta_k(t)\right| &\le \|\Delta_k\|_\infty\,\|f\|_\infty + \left|\int_X^{2X}\Delta_k(t)f'(t)dt\right|\\ &\ll \|f\|_\infty X^{1/k-1/k^2} + \left|\int_X^{2X}\Delta_k(t)f'(t)dt\right|.\end{aligned}$$

§3 Proof of Theorem 5.

By a splitting-up argument and Minkowski's inequality, it suffices to show that

$$\int_T^{2T} \left| \sum_{n\sim X} \frac{r_k(n)}{n^{\sigma+it}} \right|^2 dt \ll T^{1+\epsilon/2}. \tag{3.1}$$

The left-hand side of (3.1) is

$$\begin{aligned} &\sum_{n\sim X} \sum_{m\sim X} \frac{r_k(n)}{n^\sigma} \frac{r_k(m)}{m^\sigma} \int_T^{2T} (m/n)^{it} dt \\ &\qquad \leq 4 \sum_{n\sim X} \frac{r_k(n)}{n^\sigma} \sum_{n\leq m\leq X} \frac{r_k(m)}{m^\sigma} \min\left(T, \frac{1}{\log m/n}\right). \end{aligned} \tag{3.2}$$

The contribution to the last double sum in (2) from $m = n$ is

$$\ll T \sum_{n\sim X} \frac{r_k^2(n)}{n^{2\sigma}} \ll T,$$

since $r_k(n) \ll n^\epsilon$ and $\sum\limits_{n\sim X} \frac{r_k(n)}{n^{2\sigma}} \ll X^{2/k-2\sigma}$.

By a further splitting-up argument, it suffices to show that the contribution to the last double sum in (3.2) from $n \sim X$, $m - n \sim Y$ is $O(T^{1+\epsilon/3})$ for $\frac{1}{2} \leq Y < X$. Moreover, for $m - n \sim Y$,

$$\log \frac{m}{n} \asymp \frac{m-n}{n} \asymp \frac{Y}{X}.$$

Thus we must show that

$$X^{-2\sigma} \min\left(T, \frac{X}{Y}\right) \sum_{n\sim X} r_k(n) \sum_{m-n\sim Y} r_k(m) \ll T^{1+\epsilon/3}.$$

Now

$$\begin{aligned} \sum_{m-n\sim Y} r_k(m) &= A_k((n+2Y)^{2/k} - (n+Y)^{2/k}) \\ &\quad + \Delta_k(n+2Y) - \Delta_k(n+Y), \end{aligned}$$

and

$$X^{-2\sigma} \min\left(T, \frac{X}{Y}\right) \sum_{n\sim X} r_k(n)((n+2Y)^{2/k} - (n+Y)^{2/k})$$
$$\ll X^{-2\sigma+1}Y^{-1} \cdot X^{2/k} \cdot YX^{2/k-1} \ll T,$$

since

$$X^{4/k-2\sigma} \leq T.$$

Accordingly, it suffices to show that

$$\sum_{n\sim X} r_k(n)(G(n+2Y) - G(n+Y)) \ll T^{\epsilon/3} X^{2\sigma}\left(1 + \frac{TY}{X}\right) \tag{3.3}$$

where $G(\omega) = c_k' \omega^{1/k-1/k^2} \Phi_k(\omega^{2/k}) + B_k(\omega)$, and that

$$\sum_{n\sim X} r_k(n)(P_k((n+2Y)^{2/k}) - P_k(n+Y)^{2/k}) \ll T^{\epsilon/3} X^{2\sigma}\left(1 + \frac{TY}{X}\right). \tag{3.4}$$

Let $L = Y^{-1}X^{1-1/k}$. Then in $[X, 3X]$,

$$G(\omega) = H(\omega) + O(X^{1/k-1/k^2} L^{-1/k}),$$

with

$$H(\omega) = c_k' \omega^{1/k-1/k^2} \sum_{\ell\leq L} \ell^{-1-1/k} \cos 2\pi\left(\ell\omega^{1/k} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right).$$

(Possibly $H(\omega) = 0$.) For $\omega \in [X, 2X]$,

$$H'(\omega) \ll X^{2/k-1/k^2-1} \sum_{\ell\leq L} \ell^{-1/k} \ll X^{2/k-1/k^2-1} L^{1-1/k},$$
$$G(\omega+2Y) - G(\omega+Y) \ll X^{2/k-1/k^2-1} L^{1-1/k} Y + X^{1/k-1/k^2} L^{-1/k}$$
$$\ll Y^{1/k}.$$

Hence the left-hand side of (3.3) is

$$\ll X^{2/k} Y^{1/k}.$$

If $Y \le X/T$, then

$$X^{2/k}Y^{1/k} \ll X^{3/k}T^{-1/k} \ll X^{2\sigma},$$

since $X^{3-2k\sigma} \le T$. If $Y > X/T$, then

$$X^{2/k}Y^{1/k} \ll X^{2/k}\left(\frac{X}{T}\right)^{-(1-1/k)} Y = YX^{3/k-1}T^{1-1/k} \ll YTX^{2\sigma-1}$$

for the same reason. This proves (3.3).

Let $\psi^*(u) = \psi(u)$ for $u \notin \mathbb{Z}$, $\psi^*(u) = 1/2$ for $u \in \mathbb{Z}$. Then ψ^* is of bounded variation and continuous from the left, as is

$$P_k^*(u) = -8 \sum_{2^{-1/k}u^{1/2} \le n \le u^{1/2}} \psi^*((u^{k/2} - n^k)^{1/k}).$$

We observe that

$$P_k^*(\omega^{2/k}) - P_k(\omega^{2/k}) \ll X^{\epsilon} \quad (\omega \in [X, 2X])$$

since $\omega - n^k = m^k$, (m an integer) for $O(X^\epsilon)$ values of n in $[2^{-1/k}\omega^{1/k}, \omega^{1/k}]$. Since $\sigma > 1/k$, it suffices to prove a variant of (3.4) with P_k replaced by P_k^*; that is, to prove

$$\int_X^{2X} \{P_k^*((\omega + 2Y)^{2/k}) - P_k^*((\omega + Y)^{2/k})\} d\,T_k(\omega)$$
$$\ll T^{\epsilon/3}X^{2\sigma}\left(1 + \frac{TY}{X}\right).$$

Moreover,

$$\int_X^{2X} \{P_k^*((\omega+2Y)^{2/k}) - P_k^*((\omega+Y)^{2/k})\} d(A_k \omega^{2/k})$$
$$= \frac{2A_k}{k}\left\{ \int_{X+2Y}^{2X+2Y} P_k^*(\omega^{2/k})(\omega-2Y)^{2/k-1} d\omega \right.$$
$$\left. - \int_{X+Y}^{2X+Y} P_k^*(\omega^{2/k})(\omega-Y)^{2/k-1} d\omega \right\}$$
$$= \frac{2A_k}{k}\left\{ \int_{X+2Y}^{2X+2Y} P_k^*(\omega^{2/k})\{(\omega-2Y)^{2/k-1} - (\omega-Y)^{2/k-1}\} d\omega \right.$$
$$- \int_{X+Y}^{X+2Y} P_k^*(\omega^{2/k})(\omega-Y)^{2/k-1} d\omega$$
$$+ \int_{X+Y}^{X+2Y} P_k^*(\omega^{2/k})(\omega-Y)^{2/k-1} d\omega.$$

In the last expression, the first integral is estimated using (2.3) as

$$\ll X^{2/3k} Y X^{2/k-1} \ll X^{2\sigma-1} TY,$$

since

$$X^{4/k-2\sigma} \ll T.$$

The last two integrals are also

$$\ll X^{2/3k} Y X^{2/k-1} \ll X^{2\sigma-1} TY.$$

Thus it remains to prove

$$\int_X^{2X} P_k^*(\omega+Y_1) d\Delta_k(\omega) \ll T^{\epsilon/3} X^{2\sigma} \left(1 + \frac{TY}{X}\right) \tag{3.5}$$

for $Y_1 = Y$, $2Y$. We may suppose that

$$Y < X^{1-4/3k}. \tag{3.6}$$

For in the contrary case, the left-hand side of (3.5) can be estimated by Lemma 15(i) as

$$\begin{aligned}\ll X^{8/3k} &\ll X^{2\sigma-1+4/k-2\sigma}Y\\ &\ll X^{2\sigma-1}TY,\end{aligned}$$

since $X^{4/k-2\sigma} \le T$.

Write $\omega_1 = \omega + Y_1$ and $H_r = X^{3/k-2\sigma}2^{-rq}$. We observe that, for $\omega \in [X, 2X]$,

$$\begin{aligned}(3.7)\qquad &\left| P_k^*(\omega_1^{2/k}) + 8\sum_{r=0}^{R} \sum_{0<|h|\le H_r} a_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega_1^{2/k})} e(h(\omega_1 - n^k)^{1/k}) \right|\\ &\qquad \le \sum_{r=0}^{R} \sum_{|h|\le H_r} b_r \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega_1^{2/k})} e(h(\omega_1 - n^k)^{1/k})\\ &\qquad\quad + C(k)(\log 2T)^3,\end{aligned}$$

with $R = O(\log 2T)$. This follows from (2.10), (2.11) if $\omega_1^{2/k}$ is not an integer, and by a limiting argument otherwise. Hence Lemma 15 (ii) yields

(3.8)

$$\begin{aligned}&\int_X^{2X} P_k^*(\omega_1^{2/k}) d\Delta_k(\omega)\\ &\quad = -8\sum_{r=0}^{R} \int_X^{2X} \sum_{0<|h|\le H_r} a_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega^{2/k})} e(h(\omega_1 - n^k)^{1/k}) d\Delta_k(\omega)\\ &\qquad + O\Bigg(\left| \sum_{r=0}^{R} \int_X^{2X} \sum_{|h|\le H_r} b_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega^{2/k})} e(h(\omega_1 - n^k)^{1/k}) d\Delta_k(\omega) \right|\\ &\qquad + \left| \sum_{r=0}^{R} \int_X^{2X} \sum_{|h|\le H_r} b_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega_1^{2/k})} e(h(\omega_1 - n^k)^{1/k}) \omega^{2/k-1} d\omega \right|\\ &\qquad + \left| \int_X^{2X} (\log T)^3 d\Delta_k(\omega) \right| + \int_X^{2X} (\log T)^3 \omega^{2/k-1} d\omega \Bigg).\end{aligned}$$

The last two O-terms in (3.8) contribute

$$O(X^{2/k}(\log 2T)^3) = O(X^{2\sigma})$$

by Lemma 15 (i). The contributions from b_0 in the sums over h are both

$$O\left(\sum_{r=0}^{R} \frac{x^{3/k}2^{-rq}}{H_r}\right) = O(X^{2\sigma}\log 2T)$$

from the choice of H_r.

Fix a value of r, $0 \le r \le R$, and write $P = 2^r$. After a splitting-up argument, we see that it suffices to prove

$$H^{-1}\int_X^{2X} \sum_{h\sim H} c_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega^{2/k})} e(h(\omega_1 - n^k)^{1/k})d\Delta_k(\omega) \ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right) \tag{3.9}$$

and

$$H^{-1}\int_X^{2X} \sum_{h\sim H} c_h \sum_{N_r(\omega_1^{2/k})\le n\le N_{r+1}(\omega^{2/k})} e(h(\omega_1 - n^k)^{1/k})d(\omega^{2/k}) \ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right) \tag{3.10}$$

whenever

$$\frac{1}{2} \le H < X^{3/k-2\sigma}P^{-q}\ ,\ |c_h| \le 1.$$

Let $a = (2k-1)/(2k-2)$. Using (2.11), we write the integrands in (3.9), (3.10) as

$$P^{-a}H^{-1/2}\omega_1^{1/2k}\sum_{h\sim H}\sum_{m\in[hP,2hP]} b(h,m)e(\omega^{1/k}\,|(h,m)|) + O(\log 2T),$$

with $b(h,m) \ll 1$. We have already shown that the term $O(\log 2T)$ gives rise to an acceptable error. This reduces our task to showing that

$$P^{-a}H^{-3/2}\sum_{h\sim H}\sum_{m\in[hP,2hP]} b(h,m)\int_X^{2X} \omega_1^{1/2k}e(\omega_1^{1/k}\,|(h,m)|)d\Delta_k(\omega) \ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right) \tag{3.11}$$

and

$$P^{-a}H^{-3/2}\sum_{h\sim H}\ \sum_{m\in[hP,2hP]} b(h,m)\int_X^{2X} \omega_1^{1/2k} e(\omega_1^{1/k}|(h,m)|)\omega^{2/k-1}d\omega \tag{3.12}$$
$$\ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right).$$

The bound (3.12) gives no trouble. The integrals are

$$O(X^{5/2k-1}(|(h,m)|X^{1/k-1})^{-1}) = O(X^{3/2k}(PH)^{-1})$$

from Lemma 8. Thus the left-hand side of (3.12) is

$$\ll P^{1-a}H^{1/2}X^{3/2k}(PH)^{-1} \ll X^{3/2k} \ll X^{2\sigma}.$$

For the integrals in (3.11), we use Lemma 15 (iii):

$$\int_X^{2X} e(\omega_1^{1/k}|(h,m)|)\omega_1^{1/2k}d\Delta_k(\omega)$$
$$\ll X^{3/2k-1/k^2} + \left|\int_X^{2X} \omega_1^{(1/2k)-1} e(\omega_1^{1/k}|(h,m)|)\Delta_k(\omega)d\omega\right|$$
$$+\left|\int_X^{2X} \omega_1^{(3/2k)-1}|(h,m)|\, e(\omega_1^{1/k}|(h,m)|)\Delta_k(\omega)d\omega\right|.$$

The contribution of the first two terms in this bound to the left-hand side of (3.11) is

$$\ll P^{1-a}H^{1/2}X^{3/2k-1/k^2}$$
$$\ll X^{3/2k-\sigma+3/2k-1/k^2} \ll X^{2\sigma}.$$

Recalling (2.1) once more, it remains to show that

(3.13)

$$P^{1-a}H^{-1/2}\sum_{h\sim H}\ \sum_{m\in[hP,2hP]}\left|\int_X^{2X} \omega_1^{(3/2k)-1} e(\omega_1^{1/k}|(h,m)|)\omega^{1/k-1/k^2}\Phi_k(\omega^{2/k})d\omega\right|$$
$$\ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right)$$

and

$$P^{1-a}H^{-1/2}\sum_{h\sim H}\sum_{m\in[hP,2hP]}\left|\int_X^{2X}\omega_1^{(3/2k)-1}e(\omega_1^{1/k}|(h,m)|)P_k(\omega^{2/k})d\omega\right| \ll T^{\epsilon/4}X^{2\sigma}\left(1+\frac{TY}{X}\right). \tag{3.14}$$

For (3.13), we have the bound

$$\int_X^{2X}\omega_1^{(3/2k)-1}\omega^{1/k-1/k^2}e(\omega_1^{1/k}|(h,m)|\pm\ell\omega^{1/k})d\omega \ll X^{3/2k-1/k^2}|\ell-|(h,m)||^{-1}$$

unless

$$|\ell-|(h,m)||<C(k)\,\frac{Y}{X}\,PH+1. \tag{3.15}$$

Now

$$\sum_{|\ell-|(h,m)||\geq C(k)\,\frac{Y}{X}\,PH+1}\ell^{-1-1/k}|\ell-|(h,m)||^{-1}\ll(PH)^{-1}.$$

For the contribution from $\ell-|(h,m)|>PH$ and $\ell<\frac{|(h,m)|}{2}$ is clearly $O((PH)^{-1})$. The remaining ℓ contribute

$$O\left(\sum_{1\leq\ell'\leq PH}(PH)^{-1-1/k}(\ell')^{-1}\right)=O((PH)^{-1}).$$

We also observe that

$$\sum_{|\ell-|(h,m)||\leq C(k)\,\frac{Y}{X}\,PH+1}\ell^{-1-1/k} \ll (PH)^{-1-1/k}\left(\frac{Y}{X}\,PH+1\right).$$

Combining these estimates, we see that the integral in (3.13) is

$$\ll(PH)^{-1}X^{3/2k-1/k^2}+(PH)^{-1-1/k}\left(\frac{Y}{X}\,PH+1\right)X^{5/2k-1/k^2}.$$

The left-hand side of (3.13) is thus

$$\ll H^{1/2}X^{3/2k-1/k^2} + (PH)^{3/2-1/k}X^{5/2k-1/k^2-1}Y + H^{1/2-1/k}X^{5/2k-1/k^2}.$$

Now

$$H^{1/2}X^{3/2k-1/k^2} \ll X^{3/2k-\sigma+3/2k-1/k^2} \ll X^{2\sigma}$$

since $\sigma > 1/k$;

$$\begin{aligned} H^{1/2-1/k}X^{5/2k-1/k^2} &\ll X^{(1/2-1/k)(3/k-2\sigma)+5/2k-1/k^2} \\ &\ll X^{2\sigma} \end{aligned}$$

since $\sigma \geq (4k-4)/(3k^2-2k)$. Finally,

$$\begin{aligned} (PH)^{3/2-1/k}X^{5/2k-1/k^2-1}Y &\ll X^{(3/2-1/k)(3/k-2\sigma)+5/2k-1/k^2-1}Y \\ &\ll X^{2\sigma-1}TY \end{aligned}$$

because

$$\begin{aligned} X^{(3/2-1/k)(3/k-2\sigma)+5/2k-1/k^2-2\sigma} &= X^{7/k-4/k^2-\sigma(5-2/k)} \\ &\leq X^{4/k-2\sigma} \leq T. \end{aligned}$$

$\Big($This is a consequence of the obvious inequality

$$\sigma\left(3-\frac{2}{k}\right) > \frac{3}{k} - \frac{2}{k^2}.\Big)$$

This establishes (3.13).

Turning to (3.14), another application of (2.10) gives

(3.16)

$$\begin{aligned} &\int_X^{2X} \omega_1^{3/2k-1} P_k(\omega^{2/k}) e(\omega_1^{1/k}|(h,m)|)d\omega \\ &\quad = -8\sum_{s=0}^{R} \sum_{0<|h_1|\leq K_s} a_{h_1} \int_X^{2X} \omega_1^{3/2k-1} \sum_{n=N_s(\omega^{2/k})}^{N_{s+1}(\omega^{2/k})} e(h_1(\omega-n^k)^{1/k} + \omega_1^{1/k}|(h,m)|)d\omega \\ &\quad + O\left(\sum_{s=0}^{R}\int_X^{2X} \omega_1^{3/2k-1} \sum_{|h_1|\leq K_s} b_{h_1} \sum_{n=N_s(\omega^{2/k})}^{N_{s+1}(\omega^{2/k})} e(h_1(\omega-n^k)^{1/k})d\omega\right) \\ &\qquad + O(X^{3/2k}(\log 2T)^3). \end{aligned}$$

Here

$$K_s = P^{1-a}2^{-sq}H^{3/2}X^{5/2k-2\sigma},$$

so that

$$\begin{aligned}\int_X^{2X} \omega_1^{3/2k-1} b_0 \sum_{n=N_s}^{N_{s+1}} 1\, d\omega &\ll \frac{X^{5/2k}2^{-sq}}{K_s} \\ &\ll P^{a-1}H^{-3/2}X^{2\sigma}.\end{aligned}$$

Thus the terms arising from b_0 in (3.16) contribute to the left-hand side of (3.14) an amount

$$\ll P^{1-a}H^{3/2}P^{a-1}H^{-3/2}X^{2\sigma} \ll X^{2\sigma}.$$

The contribution arising from the term $O(X^{3/2k}(\log 2T)^3)$ in (3.16) is

$$\begin{aligned}O(PH^{3/2}X^{3/2k}(\log 2T)^3) &= O(X^{3/2(3/k-2\sigma)+3/2k}(\log 2T)^3) \\ &= O(X^{2\sigma}(\log 2T)^3),\end{aligned} \tag{3.17}$$

since

$$\sigma \geq \frac{4k-4}{3k^2-2k} \geq \frac{6}{5k} \quad (k \geq 4),\ \sigma \geq \frac{2}{5} \quad (k=3).$$

For the remaining terms on the right-hand side of (3.16), select a particular s, write $Q = 2^s$, and apply (2.11) to the sum over n, with s in place of r. Since the term $O(\log(|h|\, U + 2))$ leads to a further error $O(X^{3/2k}(\log 2T)^2)$, our task now reduces to showing that

(3.18)

$$\begin{aligned}P^{1-a}H^{-1/2}Q^{-a}K^{-3/2} \sum_{h\sim H} \sum_{Ph\leq m\leq 2Ph} \sum_{h_1\sim K} \sum_{Qh_1\leq m_1\leq 2Qh_1} |I_\delta(h,m,h_1,m_1)| \\ \ll T^{\epsilon/5}X^{2\sigma}\left(1+\frac{TY}{X}\right).\end{aligned}$$

Here $1 \leq K \leq K_s$,

$$I_\delta(h,m,h_1,m_1) = \int_X^{2X} \omega_1^{3/2k-1}\omega^{1/2k} e\left(|(h_1,m_1)|\, \omega^{1/k} + \delta\, |(h,m)|\, \omega_1^{1/k}\right) d\omega,$$

and δ may be 0, 1 or -1.

We first consider (3.18) when either $\delta = 0$ or 1, or $QK > C(k)PH$. In this case

$$\frac{d}{d\omega}\,(|(h_1, m_1)|\,\omega^{1/k} + \delta\,|(h, m)|\,\omega_1^{1/k}) \gg QKX^{1/k-1},$$

$$I_\delta(h, m, h_1, m_1) \ll X^{2/k-1}(QKX^{1/k-1})^{-1} = X^{1/k}(QK)^{-1}$$

from Lemma 8. The left-hand side of (3.18) is

$$\begin{aligned}&\ll P^{2-a}H^{3/2}Q^{1-a}K^{1/2}X^{1/k}(QK)^{-1}\\ &\ll P^{-1/2}(PH)^{3/2}(QK)^{-1/2}X^{1/k} \ll PHX^{1/k} \ll X^{2\sigma}\end{aligned}$$

as we saw in (3.17). Similarly, when $PH > C(k)QK$ we have

$$I_{-1}(h, m, h_1, m_1) \ll X^{1/k}(PH)^{-1},$$

and the left-hand side of (3.18) is

$$\begin{aligned}&\ll P^{2-a}H^{3/2}Q^{1-a}K^{1/2}X^{1/k}(PH)^{-1}\\ &\ll (PH)^{1/2}(QK)^{1/2}X^{1/k} \ll PHX^{1/k} \ll X^{2\sigma},\end{aligned}$$

as we saw in (3.17).

For the case $\delta = -1$, $QK \asymp PH$, we observe that

$$\begin{aligned}(3.19)\qquad &\frac{d}{d\omega}\,(|h, m)|\,\omega_1^{1/k} - |(h_1, m_1)|\,\omega^{1/k-1})\\ &= \frac{1}{k}\,\big|(|(h, m)| - |(h_1, m_1)|)\omega^{1/k-1}\big| + O\left(\frac{PHY}{X}\,\omega^{1/k-1}\right).\end{aligned}$$

Consider the contribution to (3.18) from quadruples with

$$(3.20)\qquad \left(\Delta - \frac{C(k)Y}{X}\right)PH < ||(h, m)| - |(h_1, m_1)|| \le 2\Delta PH.$$

where Δ runs over the $O(\log 2T)$ values

$$\Delta = 2^t\,\frac{C(k)Y}{X}\ ,\ t = 0, 1, \ldots, \Delta \ll 1.$$

It suffices to show that for each Δ, these quadruples contribute $O\left(T^{\epsilon/6}X^{2\sigma}\left(1+\frac{TY}{X}\right)\right)$ to the left-hand side of (3.18). From Lemma 10, the number of quadruples satisfying (3.20) is

$$\ll T^{\epsilon/6}(PH^2+\Delta P^3H^4)^{1/2}(QK^2+\Delta Q^3K^4)^{1/2}.$$

We now consider three cases.

Case 1. We have $\Delta < (PH)^{-2}$.

In this case the number of quadruples satisfying (3.20) is

$$\ll P^{1/2}HQ^{1/2}K \ll (PH)^2(PQ)^{-1/2}.$$

Estimating the integral trivially, the contribution to the left-hand side of (3.18) is

$$\begin{aligned}&\ll P^{1-a}Q^{-a}H^{-1/2}K^{-3/2}(PH)^2(PQ)^{-1/2}X^{2/k}\\ &\ll (PH)^{3/2}(QK)^{-3/2}X^{2/k} \ll X^{2/k}\\ &\qquad\qquad\qquad\qquad\qquad\quad \ll X^{2\sigma}.\end{aligned}$$

Case 2. We have $\Delta \geq (PH)^{-2}$ and $t=0$, that is, $\Delta = C(k)YX^{-1}$.

In this case the number of quadruples satisfying (3.20) is

$$\begin{aligned}&\ll T^{\epsilon/6}(\Delta P^3H^4)^{1/2}(\Delta Q^3K^4)^{1/2}\\ &\ll T^{\epsilon/6}\Delta(PH)^4(PQ)^{-1/2}\\ &\ll T^{\epsilon/6}YX^{-1}(PH)^4(PQ)^{-1/2}.\end{aligned}$$

Estimating the integral trivially, the contribution to the left-hand side of (3.18) is

$$\begin{aligned}&\ll T^{\epsilon/6}P^{1-a}Q^{-a}H^{-1/2}K^{-3/2}YX^{-1}(PH)^4(PQ)^{-1/2}X^{2/k}\\ &\ll T^{\epsilon/6}(PH)^{7/2}(QK)^{-3/2}YX^{2/k-1}\\ &\ll T^{\epsilon/6}(PH)^2YX^{2/k-1} \ll T^{\epsilon/6}X^{8/k-4\sigma-1}Y\\ &\qquad\qquad\qquad\qquad\qquad\quad \ll X^{2\sigma-1}YT^{1-\epsilon/6}\end{aligned}$$

since

$$X^{8/k-6\sigma} \leq X^{4/k-2\sigma} \leq T.$$

Case 3. $\Delta \geq (PH)^{-2}$ and $t \geq 1$, so that $\Delta \geq 2C(k)Y/X$.

We can now infer from (3.19) that the quadruples satisfying (3.20) have

$$\begin{aligned} I_{-1}(h,m,h_1,m_1) &\ll X^{2/k-1}(\Delta PHX^{1/k-1})^{-1} \\ &\ll X^{1/k}(PH)^{-1}\Delta. \end{aligned}$$

The number of quadruples is again $\ll T^{\epsilon/6}\Delta(PH)^4(PQ)^{-1/2}$. Thus the contribution to the left-hand side of (3.18) is

$$\begin{aligned} &\ll T^{\epsilon/6}P^{1-a}Q^{-a}H^{-1/2}K^{-3/2}\Delta(PH)^4(PQ)^{-1/2}X^{1/k}(PH)^{-1}\Delta^{-1} \\ &\ll T^{\epsilon/6}P^{-1}(PH)^{5/2}(QK)^{-3/2}X^{1/k} \\ &\ll T^{\epsilon/6}HX^{1/k} \ll T^{\epsilon}X^{2\sigma} \end{aligned}$$

from (3.17). This completes the proof of Theorem 5.

§4 Proof of Theorem 4.

Let σ_0 be fixed, $\frac{2}{5} \leq \sigma_0 \leq \frac{1}{2}$ $(k=3)$, $\frac{3}{2k} - \frac{1}{k^2} \leq \sigma_0 \leq \frac{3}{2k}$ $(k = 4, 5 \ldots)$. Define the positive number X by $X^{2\sigma_0} = T$. It is immediate from Lemma 1 that

$$M_k(\sigma_0, T) \ll W_1 + T^2 W_2 + T, \tag{4.1}$$

where

$$W_1 = \int_T^{2T} \left| \sum_{n \leq X} \frac{r_k(n)}{n^{\sigma_0+it}} \right|^2 dt,$$

$$W_2 = \int_T^{2T} \left| \int_X^{\infty} \frac{\Delta_k(\omega)}{\omega^{\sigma_0+it+1}}\, d\omega \right|^2 dt.$$

We may apply Theorem 5 to W_1; the conditions

$$\sigma_0 \geq \frac{2}{5}\,(k=3),\ \sigma_0 \geq (4k-4)/(3k^2-2k),$$

$$\max\left(\frac{4}{k} - 2\sigma_0, 3 - 2\sigma_0 k\right) \leq 2\sigma_0$$

are easily seen to be satisfied.

It follows that $W_1 \ll T^{1+\epsilon}$. Recalling the decomposition of $\Delta_k(x)$ in (2.1), we have only to show that

$$W_3 = \int_T^{2T} \left| \int_X^{\infty} \frac{\Phi_k(\omega^{2/k})}{\omega^{\sigma_1+it}} \, d\omega \right|^2 dt \ll T^{-1+\epsilon},$$

where $\sigma_1 = \sigma_0 + 1 - 1/k + 1/k^2$, and

$$W_4 = \int_T^{2T} \left| \int_X^{\infty} \frac{F_k(\omega)}{\omega^{\sigma_2+it}} \, d\omega \right|^2 dt \ll T^{-1+\epsilon},$$

where $\sigma_2 = \sigma_0 + 1$, $F_k(\omega) = P_k(\omega) + B_k(\omega^{k/2})$.

Crudely, we have

$$\int_T^{2T} \left| \int_X^{\infty} \frac{\sum_{n>B} n^{-1-1/k} \cos\left(2\pi n \omega^{1/k} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right)}{\omega^{\sigma_1+it}} \, d\omega \right|^2 dt$$

$$\ll T B^{-2/k} X^{-2\sigma_1+2} \ll T^{-1}$$

if we choose

$$B = T^k X^{k(-\sigma_1+1)}.$$

After a splitting up of the sum

$$\sum_{n \le B} n^{-1-1/k} \cos\left(2\pi n \omega^{1/k} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right),$$

we find that for some N, $\frac{1}{2} \le N < B$, we have

$$W_3 \ll T^{-1} + (\log T)^2 \int_T^{2T} \left| \int_X^{\infty} g(\omega) \omega^{-it} \right|^2 dt, \tag{4.2}$$

where

$$g(\omega) = \omega^{-\sigma_1} \sum_{n \sim N} n^{-1-1/k} \cos\left(2\pi n \omega^{1/k} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right).$$

We decompose the integral $\displaystyle\int_X^{\infty}$ as $\displaystyle\sum_{j=0}^{\infty} \int_{J(j)}$, where

$$J(j) = [X2^j, X2^{j+1}].$$

We have

(4.3)
$$\left|\int_{J(j)} g(\omega)\omega^{-it}d\omega\right| \leq \frac{1}{2}\left|\sum_{n\sim N} n^{-1-1/k}\left(\int_{J(j)} \omega^{-\sigma_1} e\left(n\omega^{1/k} - \frac{t\log\omega}{2\pi}\right)d\omega\right)\right|$$
$$+\frac{1}{2}\left|\sum_{n\sim N} n^{-1-1/k}\left(\int_{J(j)} \omega^{-\sigma_1} e\left(-n\omega^{1/k} - \frac{t\log\omega}{2\pi}\right)d\omega\right)\right|.$$

Let $F(\omega) = n\omega^{1/k} - \frac{t\log\omega}{2\pi}$ and $F_1(\omega) = -n\omega^{1/k} - \frac{t\log\omega}{2\pi}$. We have

$$|F_1'(\omega)| \gg \max\left(N(X2^j)^{1/k-1}, \frac{T}{X2^j}\right)$$

in (4.3). If

$$k^{-1}N(X2^j)^{1/k} > \frac{T}{\pi}, \tag{4.4}$$

we have

$$|F'(\omega)| \gg N(X2^j)^{1/k-1}$$

in (4.3), while if

$$2k^{-1}N(X2^{j+1})^{1/k} < \frac{T}{4\pi}, \tag{4.5}$$

we have instead

$$|F'(\omega)| \gg T(X2^j)^{-1}$$

in (4.3).

We conclude from Lemma 8 that

$$\int_{J(j)} g(\omega)\omega^{-it}d\omega \ll N^{-1-1/k}(2^jX)^{-\sigma_1-1/k+1} \tag{4.6}$$
$$\ll N^{-1/k}T^{-1}(2^jX)^{1-\sigma_1}$$

if (4.4) holds, while

$$\int_{J(j)} g(\omega)\omega^{-it}d\omega \ll N^{-1/k}T^{-1}(2^jX)^{1-\sigma_1} \tag{4.7}$$

if (4.5) holds. There are only $O(1)$ 'exceptional' values of j satisfying neither (4.4) nor (4.5). For these j,

$$N(X2^j)^{1/k} \asymp T.$$

(Of course, there are no exceptional j unless $N \ll TX^{-1/k}$.)

If there are no exceptional j, then we can apply (4.6), (4.7) as follows:

$$\begin{aligned}(4.8) \qquad & \int_T^{2T} \left| \int_X^\infty g(\omega)\omega^{-it} d\omega \right|^2 dt \\ & \leq \int_T^{2T} \left(\sum_{j=1}^\infty j^{-2} \right) \left(\sum_{j=1}^\infty j^2 \left| \int_{J(j)} g(\omega)\omega^{-it} d\omega \right|^2 dt \right) \\ & \ll \int_T^{2T} \sum_{j=1}^\infty j^2 \left| \int_{J(j)} g(\omega)\omega^{-it} d\omega \right|^2 dt \\ & \ll T^{-1} X^{2-2\sigma_1} = T^{-1} X^{-2\left(\sigma_0 - \frac{1}{k} - \frac{1}{k^2}\right)}.\end{aligned}$$

Recalling (4.2), we obtain the bound

$$W_3 \ll T^{-1}$$

using only the lower bound $\sigma_0 > \frac{1}{k} - \frac{1}{k^2}$.

Suppose now that there are exceptional j. For some fixed j_0 satisfying

$$N(X2^{j_0})^{1/k} \asymp T, \tag{4.9}$$

we can modify the above calculation to obtain

$$\begin{aligned}(4.10) \qquad & \int_T^{2T} \left| \int_X^\infty g(\omega)\omega^{-it} d\omega \right|^2 dt \\ & \ll \int_T^{2T} \left| \int_J g(\omega)\omega^{-it} d\omega \right|^2 dt \\ & \quad + T^{-1} X^{2-2\sigma_1}.\end{aligned}$$

Here $J = J(j_0)$.

We now appeal to Lemma 5. We have

$$\int_T^{2T} \left| \int_J g(\omega)\omega^{-it} d\omega \right|^2 dt \tag{4.11}$$

$$\ll (X2^{j_0})^{1-2\sigma_1} \log T \int_J \left| \sum_{n\sim N} n^{-1-1/k} e(n\omega^{1/k}) \right|^2 d\omega.$$

A change of variable shows that the integral on the right-hand side of (4.11) is

$$\int_{(2^{j_0}X)^{1/k}}^{(2^{j_0+1}X)^{1/k}} kv^{k-1} \left| \sum_{n\sim N} n^{-1-1/k} e(nv) \right|^2 dv$$

$$\ll (2^{j_0}X)^{1-1/k} N^{-(1+2/k)} (2^{j_0}X)^{1/k}$$

by Parseval's equality applied to subintervals of $[(2^{j_0}X)^{1/k}, (2^{j_0+1}X)^{1/k}]$ having length 1. We conclude that

$$\int_T^{2T} \left| \int_J g(\omega)\omega^{-it} d\omega \right|^2 dt$$

$$\ll N^{-(1+2/k)} (2^{j_0}X)^{2-2\sigma_1} \log T$$

$$\ll N^{-(1+2/k)} (T^k N^{-k})^{2-2\sigma_1} \log T$$

by (4.9). This bound is

$$\ll N^{-(1+2/k)} (T^k N^{-k})^{-1/k} \log T \ll T^{-1} \log T$$

since $\sigma_1 \geq 1 + \frac{1}{2k}$. Recalling (4.2), (4.10), we always have

$$W_3 \ll T^{-1} (\log T)^3.$$

Now we have to show that

$$W_4 \ll T^{-1+\epsilon}.$$

Arguing as in (4.8), it suffices to show that

$$\int_T^{2T} \left| \int_{J(j)} \frac{F_k(\omega)}{\omega^{\sigma_2+it}} d\omega \right|^2 dt \ll T^{-1+\epsilon} j^{-4}. \tag{4.12}$$

Lemma 5 yields, for any measurable function $E(\omega)$ such that

$$E(\omega) \ll T^{\epsilon/8} \text{ on } [X2^j, X2^{j+1}],$$

$$\int_T^{2T} \left| \int_{J(j)} \frac{E(\omega)}{\omega^{\sigma_2+it}} \, d\omega \right|^2 dt \tag{4.13}$$
$$\ll (X2^j)^{-2\sigma_2+2} (\log T) T^{\epsilon/4}$$
$$\ll (X2^j)^{-2\sigma_0} T^{\epsilon/3} \ll T^{-1+\epsilon/3} 2^{-2j\sigma_0}.$$

Thus it suffices to show that

$$\int_T^{2T} \left| \int_{J(j)} \frac{P_k(\omega^{2/k})}{\omega^{\sigma_2+it}} \, d\omega \right|^2 dt \ll T^{-1+\epsilon} j^{-4}.$$

Define R as in Section 2 with $U = (X2^j)^{k/2}$. In view of (4.13) and the decomposition (2.6), it suffices to show for a fixed r, $0 \le r \le R$, that

$$\int_T^{2T} \left| \int_{J(j)} \omega^{-\sigma_2 - it} \sum_{n=N_r(\omega^{2/k})}^{N_{r+1}(\omega^{2/k})} \psi((\omega - n^k)^{1/k}) d\omega \right|^2 dt \ll T^{-1+\epsilon/2} j^{-4}.$$

Let $H = H(T, r)$ be a positive integer, to be chosen below. Let

$$f(\omega) = \sum_{n=N_r(\omega^{2/k})}^{N_{r+1}(\omega^{2/k})} \psi((\omega - n^k)^{1/k}),$$

$$g(\omega) = -\frac{1}{2\pi i} \sum_{n=N_r(\omega^{2/k})}^{N_{r+1}(\omega^{2/k})} \sum_{0<|h|\le H} \frac{e(h(\omega - n^k)^{1/k})}{h}.$$

It will suffice to show that

$$\int_T^{2T} \left| \int_{J(j)} \omega^{-\sigma_2 - it} (f(\omega) - g(\omega)) d\omega \right|^2 dt \ll T^{-1+\epsilon/2} j^{-4}. \tag{4.14}$$

and

$$\int_T^{2T} \left| \int_{J(j)} \omega^{-\sigma_2 - it} g(\omega) d\omega \right|^2 dt \ll T^{-1+\epsilon/2} j^{-4}. \tag{4.15}$$

We begin with (4.14). Let $P = 2^r$. For $n \asymp (X2^j)^{1/k}$, we write $f_n(\omega)$ for the indicator function of the interval

$$I(n) = [n^k(1 + (2P)^{-q}),\ n^k(1 + P^{-q})].$$

Let

$$I_1(n) = [X2^j, X2^{j+1}] \cap I(n).$$

Now

$$\int_{I_1(n)} \left| \psi((\omega - n^k)^{1/k}) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(h(\omega - n^k)^{1/k})}{h} \right|^2 d\omega$$

$$= \int_{w^k + n^k \in I_1(n)} k w^{k-1} \left| \psi(w) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(hw)}{h} \right|^2 dw$$

$$\ll X2^j P^{-q} \int_0^1 \left| \psi(w) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(hw)}{h} \right|^2 dw$$

$$\ll X2^j P^{-q} H^{-1}$$

by (2.7). (Note that the variable w introduced by the change of variable satisfies $w \asymp nP^{-q/k}$.) Hence

$$\int_{J(j)} |f(\omega) - g(\omega)|^2 d\omega$$

$$= \int_{J(j)} \left| \sum_{n \asymp (X2^j)^{1/k}} f_n(\omega) \left\{ \psi((\omega - n^k)^{1/k}) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(h(\omega - n^k)^{1/k})}{n} \right\} \right|^2 d\omega$$

$$\ll (X2^j)^{1/k} \sum_{n \asymp (X2^j)^{1/k}} \int_{J(j)} f_n(\omega) \left| \psi((\omega - n^k)^{1/k}) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(h(\omega - n^k)^{1/k})}{h} \right|^2 d\omega$$

(by Cauchy's inequality)

$$= (X2^j)^{1/k} \sum_{n \asymp (X2^j)^{1/k}} \int_{I_1(n)} \left| \psi((\omega - n^k)^{1/k}) + \frac{1}{2\pi i} \sum_{0<|h|\leq H} \frac{e(h(\omega^k - n^k)^{1/k})}{h} \right|^2 d\omega$$

$$\ll (X2^j)^{1+2/k} P^{-q} H^{-1}.$$

In view of Lemma 5, the left-hand side of (4.14) is

$$\ll (X2^j)^{1-2\sigma_2} \log T \int_{I_1(n)} |f(\omega) - g(\omega)|^2 d\omega$$

$$\ll (X2^j)^{-2\sigma_0+2/k} (\log T) P^{-q} H^{-1} \ll T^{-1} \log T \, 2^{-j(2\sigma_0-2/k)}$$

if we now specify that

$$H = TX^{-2\sigma_0+2/k} P^{-q} = X^{2/k} P^{-q}.$$

Turning to (4.15), it is sufficient to show for a fixed K, $\frac{1}{2} \le K \le H$, that

$$\int_T^{2T} \left| \int_{J(j)} \omega^{-\sigma_2+\alpha it} g_K(\omega) d\omega \right|^2 dt \ll T^{-1+\epsilon/3} j^{-4},$$

where $\alpha = 1$ or -1 and

$$g_K(\omega) = \sum_{h \sim K} \sum_{n=N_r(\omega^{2/k})}^{N_{r+1}(\omega^{2/k})} e(h(\omega - n^k)^{1/k}).$$

Recalling (4.13), we can reduce this to showing that

$$\int_T^{2T} \left| \int_{J(j)} P^{-a} K^{-3/2} \omega^{-\sigma_3-\alpha it} G(\omega) d\omega \right|^2 dt \ll T^{-1+\epsilon/3} j^{-4}.$$

Here

$$\sigma_3 = \sigma_2 - 1/2k = \sigma_0 + 1 - 1/2k$$

and

$$G(\omega) = \sum_{(h,m)\in\mathcal{E}} b(h,m) e(\omega^{1/k} |(h,m)|)$$

with $b(h,m) \ll 1$,

$$\mathcal{E} = \{(h,m) : h \sim K, \ Ph \le m \le 2Ph\}.$$

Compare the reduction of (3.10) to (3.11).

Arguing as in the discussion of W_3, we find that after excluding $O(1)$ 'exceptional' values of j for which

$$\frac{c_1(k)T}{(X2^j)^{1/k}} < PK < \frac{c_2(k)T}{(X2^j)^{1/k}}, \tag{4.16}$$

(with $c_1(k) > 0$) we have

$$\int_{J(j)} \frac{e(|(h,m)|\omega^{1/k})}{\omega^{\sigma_3+\alpha it}}\, d\omega$$
$$\ll \min\big((X2^j)^{1-\sigma_3-1/k}(PK)^{-1}, (X2^j)^{1-\sigma_3}T^{-1}\big)$$

for all $(h,m) \in \mathcal{E}$.

Since $|\mathcal{E}| \ll PK^2$, the 'non-exceptional' j satisfy

$$(4.17)\quad P^{-2a}K^{-3}\int_T^{2T}\left|\int_{J(j)} \frac{G(\omega)}{\omega^{\sigma_3+\alpha it}}\, d\omega\right|^2 dt$$
$$\ll P^{-2a}K^{-3}T(PK^2)^2 \min((X2^j)^{2-2/k-2\sigma_3}P^{-2}K^{-2},$$
$$(X2^j)^{2-2\sigma_3}T^{-2})$$
$$\ll P^{-(2a-1)}\min((PK)^{-1}T(X2^j)^{2-2/k-2\sigma_3}, PKT^{-1}(X2^j)^{2-2\sigma_3})$$
$$\ll (X2^j)^{2-2\sigma_2} \ll T^{-1}2^{-2\sigma_0 j}.$$

Now suppose that j satisfies (4.16). We have

(4.18)
$$\int_{J(j)}\left|\frac{G(\omega)}{\omega^{\sigma_3}}\right|^2 d\omega$$
$$= \sum_{(h_1,m_1)\in\mathcal{E}}\ \sum_{(h_2,m_2)\in\mathcal{E}} b(h_1,m_1)\,\overline{b(h_2,m_2)}\int_{J(j)}\omega^{-2\sigma_3}e\left((|(h_1,m_1)|-|(h_2,m_2)|)\omega^{1/k}\right)d\omega$$
$$\ll \sum_{(h_1,m_1)\in\mathcal{E}}\ \sum_{(h_2,m_2)\in\mathcal{E}} \min\left((X2^j)^{1-2\sigma_3}\frac{(X2^j)^{1-2\sigma_2}}{\big||(h_1,m_1)|-|(h_2,m_2)|\big|}\right),$$

by Lemma 8.

We consider the contribution to the last sum from h_1, m_1, h_2, m_2 satisfying

$$(4.19)\qquad \Delta PK - (X2^j)^{-1/k} \le \big||(h_1,m_1)|-|(h_2,m_2)|\big| < 2\Delta PK.$$

Here Δ runs over the numbers in $(0,3]$ of the form

$$\Delta = (PK)^{-1}(X2^j)^{-1/k}2^h \quad (h = 0,1,\ldots).$$

Clearly (4.19) implies

$$|h_1^q + m_1^q - h_2^q - m_2^q| \ll \Delta(PK)^q.$$

By Lemma 11, the number $\mathcal{N}$ of such quadruples satisfies

$$\mathcal{N} \ll (PK)^{\epsilon/5}(PK^2 + \Delta P^3 K^4). \tag{4.20}$$

The contribution of these quadruples to the last sum in (4.18) is

$$\begin{aligned} &\ll (PK)^{\epsilon/5}\left\{(X2^j)^{1-2\sigma_3}PK^2 + \frac{(X2^j)^{1-2\sigma_2}}{\Delta PK}\,\Delta P^3K^4\right\} \\ &= (PK)^{\epsilon/5}\{(X2^j)^{1-2\sigma_3}PK^2 + (X2^j)^{1-2\sigma_2}P^2K^3\}. \end{aligned}$$

Summing over $O(j + \log T)$ values of Δ, we find that

$$\int_{J(j)} \left|\frac{G(\omega)}{\omega^{\sigma_3}}\right|^2 d\omega \ll jT^{\epsilon/4}\{(X2^j)^{1-2\sigma_3}PK^2 + (X2^j)^{1-2\sigma_3}P^2K^3\}.$$

Applying Lemma 5, and recalling (4.16), we have

$$\begin{aligned} P^{-2a}K^{-3}\int_T^{2T}\left|\int_{J(j)} \frac{G(\omega)}{\omega^{\sigma_3+\alpha it}}\,d\omega\right|^2 dt \tag{4.21} \\ \ll jT^{\epsilon/3}\{(X2^j)^{2-2\sigma_3}(PK)^{-1} + (X2^j)^{2-2\sigma_2}\} \\ \ll jT^{\epsilon/3}\{(X2^j)^{2-2\sigma_3+1/k}T^{-1} + (X2^j)^{-2\sigma_0}\}. \end{aligned}$$

We recall that $X^{2\sigma_0} = T$ and that

$$2 - 2\sigma_3 + 1/k = -2\sigma_0 + 2/k < 0.$$

Thus the left-hand side of (4.21) is

$$\ll jT^{-1+\epsilon/3}2^{-(2\sigma_0-2/k)j}.$$

Combining this with (4.17), we see that the proof of (4.15) is complete. As already noted before (4.14), this finishes the proof of Theorem 4.

§5 Proof of Theorem 3.

It suffices to show that

$$\int_X^{2X} E_k(x)^2 dx = d_k((2X)^{1+2/k-2/k^2} - X^{1-2/k-2/k^2}) \\ + O(X^{1+2/k-2/k^2-\eta})$$

for large X. We write (in this section only) $\| \dots \|$ for the L^2 norm on $[X, 2X]$. We note that

$$\|F + G\|^2 = \|F\|^2 + O(\|F\| \, \|G\|)$$

if $\|G\| = O(\|F\|)$. Accordingly it suffices to write $E_k(x)$ in the form

$$E_k(x) = F(x) + G_1(x) + \cdots + G_4(x), \tag{5.1}$$

and to show that

$$\|F\|^2 = d_k((2X)^{1+2/k-2/k^2} - X^{1+2/k-2/k^2}) \\ + O(X^{1+2/k-2/k^2-\eta}), \tag{5.2}$$

and that each G_j satisfies

$$\|G_j\|^2 = O(X^{1+2/k-2/k^2-2\eta}). \tag{5.3}$$

Let $\lambda = 1/k - 1/k^2 + \epsilon$. Let $c = c(\chi, k)$ be a small positive constant, $c < 1/(2k+1)$, and let $y = X^c$. By combining Lemma 2 with (2.1), we obtain

$$E_k(x) = c_k' x^{1/k-1/k^2} \sum_{d \le y} \mu(d) \Phi_k\left(\frac{x^{2/k}}{d^2}\right) + \sum_{d \le y} \mu(d) P_k\left(\frac{x^{2/k}}{d^2}\right) \\ + \frac{1}{2\pi i} \int_{\lambda - ix^C}^{\lambda + ix^C} f(y, ks) Z_k(s) \frac{x^s}{s} \, ds + O(y).$$

Thus in (5.1), we may choose

$$F(x) = c_k' x^{1/k-1/k^2} \sum_{\ell \le X} \ell^{-1-1/k} \sum_{d \le y} \frac{\mu(d)}{d^{1-1/k}} \cos 2\pi \left(\frac{\ell x^{1/k}}{d} - \frac{1}{4}\left(1+\frac{1}{k}\right)\right),$$

$$G_1(x) = c_k' x^{1/k-1/k^2} \sum_{\ell > X} \ell^{-1-1/k} \sum_{d \le y} \frac{\mu(d)}{d^{1-1/k}} \cos 2\pi \left(\frac{\ell x^{1/k}}{d} - \frac{1}{4}\left(1+\frac{1}{k}\right)\right)$$

$$= O(1),$$

$$G_2(x) = \sum_{d \le y} \mu(d) P_k\left(\frac{x^{2/k}}{d^2}\right),$$

$$G_3(x) = \int_{\lambda - ix^C}^{\lambda + ix^C} f(y, ks) Z_k(s) \frac{x^s}{s}\, ds,$$

$$G_4(x) = O(y).$$

Obviously G_1, G_4 satisfy (5.3) for $\eta \le 1/k - 1/k^2 - c$. It remains to prove (5.2), and (5.3) for G_2, G_3.

We can dismiss G_2 quickly. By Cauchy's inequality, a change of variable, and (2.4),

$$\int_X^{2X} G_2(x)^2 dx \le y \sum_{d \le y} \int_X^{2X} P_k\left(\frac{x^{2/k}}{d^2}\right)^2 dx$$

$$\ll y \sum_{d \le y} d^k \left(\frac{X^{2/k}}{d^2}\right)^{3/2+(k/2-1)} = yX^{1+1/k} \sum_{d \le y} \frac{1}{d}$$

$$\ll X^{1+1/k} y \log y.$$

Thus G_2 satisfies (5.3) for $2\eta < 1/k - 2/k^2 - c$.

For G_3, we note that, with T running over powers of 2, $2 \le T \le (2X)^C$,

$$G_3(x) \ll x^{(k-1)/k^2+\epsilon} \left(1 + \sum_T \left| \int_T^{2T} g(t) x^{it} dt \right| \right). \tag{5.4}$$

Here

$$g(t) = \frac{f(y, k\lambda + kit) Z_k(\lambda + it)}{\lambda + it}.$$

By Lemmas 7 and 4,

$$\int_X^{2X} \left| \int_T^{2T} g(t) x^{it} dt \right|^2 dx \ll X \log T \int_T^{2T} |g(t)|^2 dt \tag{5.5}$$

$$\ll \frac{X^{1+\epsilon}}{T^2} y^{2(\chi - k\lambda)} \int_T^{2T} |Z_k(\lambda + it)|^2 dt.$$

Recalling Lemma 6, the right-hand side of (5.5) is

$$\ll X^{1+2\epsilon} y^{2(\chi - k\lambda)}.$$

Combining (5.4), (5.5),

$$\int_X^{2X} G_3(x)^2 dx \ll X^{1+2/k-2/k^2+2c(\chi-k\lambda)+5\epsilon}.$$

Since $2c(\chi - k\lambda) = 2c\left(\chi - 1 + \frac{1}{k} - k\epsilon\right)$, we see that (5.3) is valid for G_3 if $\eta < c\left(1 - \frac{1}{k} - \chi\right)$.

Our treatment of $F(x)$ resembles that of Zhai [19], but we give the details for the convenience of the reader. Using the identity

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B),$$

we can write

$$2F(x)^2 = c_k'^2 x^{2/k - 2/k^2} (K_X + F_1(x) + F_2(x)),$$

where

$$K_X = \sum_{\substack{\ell_1, \ell_2 \le X, d_1, d_2 \le y \\ \ell_1 d_2 = \ell_2 d_1}} \frac{\mu(d_1)\mu(d_2)}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}},$$

$$F_1(x) = \sum_{\substack{\ell_1, \ell_2 \le X, d_1, d_2 \le y \\ \ell_1 d_2 \ne \ell_2 d_1}} \frac{\mu(d_1)\mu(d_2)}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}} \cos 2\pi \left(\left(\frac{\ell_1}{d_1} - \frac{\ell_2}{d_2} \right) x^{1/k} \right)$$

and

$$F_2(x) = \sum_{\ell_1, \ell_2 \le X, d_1, d_2 \le y} \frac{\mu(d_1)\mu(d_2)}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}} \cos 2\pi \left(\left(\frac{\ell_1}{d_1} + \frac{\ell_2}{d_2} \right) x^{1/k} - \frac{1}{2} \left(1 + \frac{1}{k} \right) \right).$$

We now apply the particular case

$$\int_X^{2X} x^{2/k-2/k^2} \cos 2\pi(\Delta x^{1/k} + a) dx \ll X^{1+1/k-2/k^2} |\Delta|^{-1}$$

of Lemma 8. This yields

$$2\int_X^{2X} F(x)^2 dx = \frac{c_k'^2 K_X}{1+2/k-2/k^2} \left((2X)^{1+2/k-2/k^2} - X^{1+2/k-2/k^2} \right) + O\left(X^{1+1/k-2/k^2} (S_1 + S_2) \right). \tag{5.6}$$

Here

$$S_1 = \sum_{\substack{\ell_1, \ell_2 \le X \\ d_1, d_2 \le y \\ \ell_1 d_2 \ne \ell_2 d_1}} \frac{1}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}} \left| \frac{\ell_1}{d_1} - \frac{\ell_2}{d_2} \right|^{-1},$$

$$S_2 = \sum_{\substack{\ell_1, \ell_2 \le X \\ d_1, d_2 \le y}} \frac{1}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}} \left(\frac{\ell_1}{d_1} + \frac{\ell_2}{d_2} \right)^{-1}.$$

We evaluate K_X as follows. We may write

$$K_X = \sum_{n \ge 1} b(n),$$

where

$$b(n) = \sum_{\substack{n = \ell_1 d_2 = \ell_2 d_1 \\ \ell_1, \ell_2 \le X,\, d_1, d_2 \le y}} \frac{\mu(d_1)\mu(d_2)}{(\ell_1 \ell_2)^{1+1/k} (d_1 d_2)^{1-1/k}}.$$

If $n \le y$, then clearly

$$\begin{aligned} b(n) &= \sum_{d_1 \mid n} \sum_{d_2 \mid n} \mu(d_1)\mu(d_2) \left(\frac{n^2}{d_1 d_2} \right)^{-1-1/k} (d_1 d_2)^{1/k-1} \\ &= n^{-2-2/k} \left(\sum_{d \mid n} \mu(d) d^{2/k} \right)^2. \end{aligned}$$

A similar calculation yields

$$|b(n)| \leq n^{-2-2/k} \left(\sum_{d \,|\, n} d^{2/k} \right)^2 \leq n^{-2+2/k} d^2(n)$$

for all n. Hence.

$$\begin{aligned} K_X &= \sum_{n=1}^{\infty} n^{-2-2/k} \left(\sum_{d \,|\, n} \mu(d) d^{2/k} \right)^2 + O\left(\sum_{n>y} n^{-2+2/k} d^2(n) \right) \\ &= e_k + O(y^{-1+2/k} X^{\epsilon}). \end{aligned} \tag{5.7}$$

Turning to S_1, it is clear that

$$\begin{aligned} S_1 &\leq y^{2/k} \sum_{\substack{\ell_1,\ell_2 \leq X \\ d_1,d_2 \leq y \\ \ell_1 d_2 \neq \ell_2 d_1}} (\ell_1 \ell_2)^{-1-1/k} |\ell_1 d_2 - \ell_2 d_1|^{-1} \\ &\ll y^{2/k} \sum_{\substack{\ell_1,\ell_2,d_1,r \\ d_1 \leq y,\, r \leq Xy}} (\ell_1 \ell_2)^{-1-1/k} r^{-1} \\ &\ll y^{2/k+1} \log X. \end{aligned}$$

Similar reasoning yields the same bound for S_2.

Recalling (5.6) and (5.7),

$$\begin{aligned} \int_X^{2X} F(x)^2 dx = d_k \left((2X)^{1+2/k-2/k^2} - X^{1+2/k-2/k^2} \right) \\ + O(X^{1+2/k-2/k^2+\epsilon} y^{-1+2/k} + X^{1+1/k-2/k^2+\epsilon} y^{1+2/k}). \end{aligned}$$

Thus (5.2) is satisfied provided that

$$\begin{aligned} &\eta < c\left(1 - \frac{2}{k}\right) \text{ and} \\ &\eta < 1/k - c(1+2/k). \end{aligned}$$

As noted above, this completes the proof of Theorem 3.

§6 Proofs of Theorems 1 and 2.

Let $k = 3$ or 4, $\sigma_3 = \frac{2}{5}$, $\sigma_4 = \frac{5}{16}$. Just as in §5, we find that

$$E_k(x) = \frac{1}{2\pi i}\int_{\sigma_k - ix^C}^{\sigma_k + ix^C} f(y_k, s) Z_k(s)\,\frac{x^s}{s}\,ds + \sum_{d\le y_k}\mu(d)\Delta_k\left(\frac{x}{d^k}\right) + O(1). \tag{6.1}$$

Here y_k is to be specified below, $y_k > 1$. Moreover, we can argue as in §5 to obtain a large T, $T \le x^C$ with

$$\begin{aligned}\int_{\sigma_k - ix^C}^{\sigma_k + ix^C} f(y_k, s) Z_k(s)\,\frac{x^s}{s}\,ds &\ll (\log x) x^{\sigma_k}\max_{|t|\le x^C} |f(y_k, k\sigma_k + it)| \\ &\qquad \left(T^{-1}\int_T^{2T} |Z_k(\sigma_k + it)|\,dt + 1\right) \\ &\ll x^{\sigma_k + \epsilon} y_k^{\rho_k - k\sigma_k}.\end{aligned} \tag{6.2}$$

We used Theorem 4 (together with the Cauchy-Schwarz inequality) and Lemma 4 in the last step.

We take

$$y_3 = x^{6\theta_3 - 4/3} = x^{0.2260\ldots},\ \ y_4 = x^{4\theta_4/3 - 1/12} = x^{0.1931\ldots}.$$

It is easily verified that in each case, (6.2) yields

$$\int_{\sigma_k - ix^C}^{\sigma_k + ix^C} f(y_k, s) Z_k(s)\,\frac{x^s}{s}\,ds \ll x^{\theta_k + \epsilon}. \tag{6.3}$$

From (2.1),

$$\sum_{d\le y_k}\mu(d)\Delta_k\left(\frac{x}{d^k}\right) = X_k + Y_k + O(y_k), \tag{6.4}$$

where

$$X_k = c_k' x^{1/k - 1/k^2}\sum_{d\le y}\frac{\mu(d)}{d^{1-1/k}}\sum_{\ell=1}^{\infty}\ell^{-1-1/k}\cos\left(\frac{2\pi\ell x^{1/k}}{d} - \frac{1}{4}\left(1 + \frac{1}{k}\right)\right)$$

and

$$Y_k = -8 \sum_{d \le y} \mu(d) \sum_{\frac{x}{2d^k} \le n^k \le \frac{x}{d^k}} \psi\left(\left(\frac{x}{d^k} - n^k\right)^{1/k}\right).$$

Clearly, in bounding X_k we need only show that for $\frac{1}{2} \le D \le y_k$, and $\ell \ge 1$, we have

$$\sum_{d \sim D} \mu(d) e\left(\frac{\ell x^{1/k}}{d}\right) \ll \ell^{1/k} D^{1-1/k} x^{\theta_k - 1/k + 1/k^2 + \epsilon}. \tag{6.5}$$

Since (6.5) is trivial for $D \le x^{k\theta_k - 1 + 1/k}$, we assume that

$$D > x^{k\theta_k - 1 + 1/k} \tag{6.6}$$

in proving (6.5). By appealing to [2], §7, we can suppose much more when $k = 3$, namely

$$D > x^{0.221}. \tag{6.7}$$

For Y_k, we follow the initial stages of the argument of Zhai and Cao, with their (2.10) as the point of departure. Let $P > 1$ and write

$$H = \max(x^{1/k - \theta_k} P^{-q}, 1).$$

We find as in (5.7), (5.9) of [20] that in bounding Y_k, it is sufficient to prove

$$\frac{x^{1/(2k)}}{P^{(1+q)/2} D^{1/2} K^{3/2}} \sum_{d \sim D} \mu(d) \left(\frac{D}{d}\right)^{1/2} \sum_{(h,\ell) \in \mathcal{E}} b(h,\ell) e\left(-\frac{x^{1/k} |(h,\ell)|}{d}\right) \ll x^{\theta_k + \epsilon} \tag{6.8}$$

for $1 \le K \le H$ and $1 \le D \le y_k$. Here

$$\mathcal{E} = \mathcal{E}(K,P) = \{(h,\ell) : h \sim K,\ Ph \le \ell \le 2Ph\}$$

as in Section 4, and $|b(h,\ell)| \le 1$. Strictly speaking, one also needs to prove the analogue of (6.8) with $\mu(d)$ replaced by 1; this is easier and need not be discussed separately.

Naturally we may apply Lemma 12. Thus we need only prove in place of (6.5) that, for a suitable U, $0 < U \leq D^{1/3}$,

$$S_I = \sum_{\substack{m\sim M \\ mn\sim D}} \sum_{n\sim N} a_m e\left(\frac{\ell x^{1/k}}{mn}\right) \ll \ell^{1/k} D^{1-1/k} x^{\theta_k - 1/k + 1/k^2 + \epsilon} \tag{6.9}$$

for $\ell \geq 1$, $MN \asymp D$, $N \gg DU^{-1}$, $|a_m| \leq 1$; and that

$$S_{II} = \sum_{\substack{m\sim M \\ mn\sim D}} \sum_{n\sim N} a_m c_n e\left(\frac{\ell x^{1/k}}{mn}\right) \ll \ell^{1/k} D^{1-1/k} x^{\theta_k - 1/k + 1/k^2 + \epsilon} \tag{6.10}$$

for $\ell \geq 1$, $MN \asymp D$, $U \ll N \ll D^{1/2}$, $|a_m| \leq 1$, $|c_n| \leq 1$.

It turns out that (6.8) requires no new work in the case $k = 4$. It is shown in §5 of Zhai and Cao [20] that

$$Y_4 \ll x^{\epsilon}(y_4 + x^{1/7} y_4^{9/28} + x^{1/8} y_4^{5/12} + x^{1/6} y_4^{1/9} + x^{0.1875}),$$

which is easily seen to be stronger than we need. In the case $k = 3$, we can quote the result we need from [2], §6 when $D < x^{2/9}$. Thus we suppose that

$$x^{2/9} < D \leq y_3 \tag{6.11}$$

in proving (6.8) for $k = 3$. Appealing to Lemma 12, we need only prove in place of (6.8) that, for a suitable U, $0 < U \leq D^{1/3}$,

$$\frac{x^{1/6}}{P^{5/4} D^{1/2} K^{3/2}} \sum_{\substack{m\sim M \\ mn\sim D}} \sum_{n\sim N} a_m \left(\frac{D}{mn}\right)^{1/2} \sum_{(h,\ell)\in\mathcal{E}} b(h,\ell) e\left(\frac{-x^{1/3}|(h,\ell)|}{mn}\right) \ll x^{\theta_3+\epsilon} \tag{6.12}$$

whenever $MN \asymp D$, $N \gg DU^{-1}$ and $|a_m| \leq 1$; and that

$$\frac{x^{1/6}}{P^{5/4} D^{1/2} K^{3/2}} \sum_{\substack{m\sim M \\ mn\sim D}} \sum_{n\sim N} a_m c_n \sum_{(h,\ell)\in\mathcal{E}} b(h,\ell) e\left(\frac{-x^{1/3}|(h,\ell)|}{mn}\right) \ll x^{\theta_3+\epsilon} \tag{6.13}$$

whenever $MN \asymp D$, $U \ll N \ll D^{1/2}$ and $|a_m| \leq 1$, $|b_n| \leq 1$.

We begin with (6.9), (6.10) for $k = 3$. We take $U = D^{1/3}$. By Lemma 13 with $X \asymp \ell x^{1/3} D^{-1}$, $N_0 = M$ and $(\kappa, \lambda) = \left(\frac{2}{7}, \frac{4}{7}\right)$, the left-hand side of (6.9) is

$$\ll (\log x)^2 \{DN^{-1/2} + D^2 x^{-1/3} + (D^{3+2\kappa} \ell^{1+2\kappa} x^{(1+2\kappa)/3} N^{-(1+2\kappa)} M^{2(\lambda-\kappa)})^{1/(6+4\kappa)}\}$$
$$\ll (\log x)^2 \{D^{2/3} + (\ell^{11} D^{25} x^{11/3} N^{-11} M^4)^{1/50}\}.$$

Moreover,

$$(D^{25} x^{11/3} N^{-11} M^4)^{1/50} < D^{19/50} x^{11/150} < D^{2/3} x^{\theta_3 - 2/9}$$

as a consequence of (6.7).

For (6.10), we appeal to Lemma 14 with $X \asymp \ell x^{1/4} D^{-1}$, $(\kappa, \lambda) = (1/2, 1/2)$, obtaining

(6.14)
$$\begin{aligned} S_{II} \ll (\log x)^3 \ell^{1/5} \{ & x^{1/15} D^{9/20} N^{1/10} + x^{1/18} D^{1/2} N^{1/9} + x^{1/15} D^{2/5} N^{1/5} \\ & + x^{1/33} D^{6/11} N^{3/11} + D^{2/3} N^{5/18} + DN^{-1/2} \\ & + x^{-1/66} D^{15/22} N^{9/22} + D^{3/2} x^{-1/6}\} \\ \ll (\log x)^3 \ell^{1/5} \{ & x^{1/15} D^{1/2} + x^{1/18} D^{5/9} + x^{1/33} D^{15/22} \\ & + D^{5/6} + x^{-1/66} D^{39/44} + D^{3/2} x^{-1/6}\}. \end{aligned}$$

Now

$$D^{5/6} \le D^{2/3} x^{\theta_3 - 2/9},$$

because $D \le y_3 = x^{6\theta_3 - 4/3}$. The remaining terms in the last expression in (6.14) are easily seen to be of smaller order than $\ell^{1/5} D^{2/3} x^{\theta_3 - 2/9}$. This establishes (6.10) and completes the proof of (6.5) for $k = 3$.

Turning to (6.9), (6.10) for $k = 4$, we suppose that

$$x^{0.0795\ldots} = x^{4\theta_4 - 3/4} < D \le y_4.$$

Let $U = D^{1/2} x^{-2\theta_4 + 3/8}$. It is easily verified that $1 \le U \le D^{1/3}$. According to Lemma 13 (i) with $(\kappa, \lambda) = (1/14, 11/14)$, $X \asymp \ell x^{1/4} D^{-1}$, $N_0 = M$, we have

$$(\log x)^{-2} S_I \ll DN^{-1/2} + D^2 x^{-1/4} + (D^{64} \ell^{16} x^4 N^{-36})^{1/88}.$$

Now

$$\begin{aligned} DN^{-1/2} &\ll D^{2/3} \ll D^{3/4} x^{\theta_4 - 3/16}, \\ D^2 x^{-1/4} &\ll D^{3/4} x^{\theta_4 - 3/16}. \end{aligned}$$

Finally,

$$(D^{64}x^4N^{-36})^{1/88} \ll (D^{28}x^4U^{36})^{1/88}$$
$$= (D^{46}x^{35/2-72\theta_4})^{1/88} \ll D^{3/4}x^{\theta_4-3/16}$$

since

$$D > x^{4\theta_4-3/4} > x^{17/10-8\theta_4}.$$

This proves (6.9).

To obtain (6.10) in the range

$$x^{4\theta_4-3/4} < D \le x^{0.125},$$

we apply Lemma 13 (ii) with $X \asymp \ell x^{1/4}D^{-1}$, $(\kappa,\lambda) = \left(\frac{89}{570}, \frac{1}{2} + \frac{89}{570}\right)$. This exponent pair is due to Huxley [5], [6]; his significantly deeper work in [8] hardly affects the value of θ_4. The condition $X \gg D$ follows from $D \le x^{0.125}$. We have

$$(\log x)^{-7/4}S_{II} \ll DN^{-1/2} + DM^{-1/4} + (D^{10+8\kappa}\ell^{1+2\kappa}x^{(1+2\kappa)/4}N)^{1/(14+12\kappa)}.$$

Now

$$DN^{-1/2} \ll DU^{-1/2} = D^{3/4}x^{\theta_4-3/16}, \tag{6.15}$$
$$DM^{-1/4} \ll D^{7/8} \ll D^{3/4}x^{\theta_4-3/16}$$

since $D \le x^{0.125} < x^{8\theta_4-3/2}$. Finally

$$(D^{10+8\kappa}x^{(1+2\kappa)/4}N)^{1/(14+12\kappa)} = (D^{6412}x^{187}N^{570})^{1/9048}$$
$$\ll (D^{6697}x^{187})^{1/9048} \ll D^{3/4}x^{\theta_4-3/16}$$

since

$$D^{89} \ge x^{89(4\theta_4-3/4)} = x^{3767/2-9048\theta_4}. \tag{6.16}$$

This is where the precise value of θ_4 arises.

It remains to obtain (6.9) for

$$x^{0.125} < D \le y_4. \tag{6.17}$$

According to Lemma 14, with $X \asymp \ell x^{1/4} D^{-1}$, $(\kappa, \lambda) = (1/2, 1/2)$, we have

$$\begin{aligned}
(6.18) \qquad (\log x)^{-3} S_{II} \ll\ & \ell^{1/5} \{x^{1/20} D^{9/20} N^{1/10} + x^{1/24} D^{1/2} N^{1/9} \\
& + x^{1/20} D^{2/5} N^{1/5} + x^{1/44} D^{6/11} N^{3/11} + D^{2/3} N^{5/18} \\
& + D N^{-1/2} + x^{-1/88} D^{15/22} N^{9/22} + D^{3/2} x^{-1/8}\} \\
\ll\ & \ell^{1/5} \{x^{1/20} D^{1/2} + x^{1/24} D^{5/9} + x^{1/44} D^{15/22} \\
& + x^{-1/88} D^{39/44} + D^{3/2} x^{-1/8} + D^{3/4} x^{\theta_4 - 3/16}\},
\end{aligned}$$

where we have applied (6.15) in the last step. Now since $D \le y_4 = x^{4\theta_4/3 - 1/12}$, we have

$$D^{3/2} x^{-1/8} \le D^{3/4} x^{\theta_4 - 3/16}.$$

Moreover,

$$x^{1/24} D^{5/9} < D^{3/4} x^{\theta_4 - 3/16},$$

since

$$D > x^{0.125} > x^{33/28 - 36\theta_4/7}.$$

It is easily verified that the remaining three terms in the last expression in (6.18) are smaller than $\ell^{1/5} D^{3/4} x^{\theta_4 - 3/16}$. This completes the proof of (6.10), and indeed (6.5), for $k = 4$. Since we already have (6.3), (6.5) and (6.8), we have finished the proof of Theorem 2.

It remains only to prove (6.12) and (6.13) for the short range (6.11) of D. If $H = 1$, then we can argue as on pp. 137–138 of Baker [2] to obtain (6.12), (6.13). Thus we suppose that $H \ge K \ge 1$, and it follows that

$$K P^{3/2} \le x^{1/3 - \theta_3}$$

from the choice of H.

We can dispose of the case

$$K \ge D P^{-1/2} x^{-5/27}$$

by repeating verbatim the argument in the last paragraph of [2], §6. We suppose that

$$(6.19) \qquad K < D P^{-1/2} x^{-5/27}.$$

We shall prove (6.12), (6.13) with

$$(6.20) \qquad U = D x^{-2\theta_3 + 1/3} P^{-1/2};$$

obviously $U < D^{1/3}$.

We can easily dispose of (6.12) using the Kusmin-Landau theorem. For if $N \gg DU^{-1} = P^{1/2}x^{2\theta_3-1/3}$, then

$$\begin{aligned} \frac{d}{dn}\left(\frac{x^{1/3}|(h,\ell)|}{mn}\right) &\asymp \frac{x^{1/3}PK}{DN} \\ &\ll x^{-2\theta_3+2/3}P^{1/2}KD^{-1} \\ &\ll x^{-2\theta_3+13/27} = x^{-0.03\ldots} \end{aligned}$$

from (6.19). A partial summation gives

$$\sum_{\substack{n\sim N \\ mn\sim D}} \left(\frac{D}{mn}\right)^{1/3} e\left(\frac{-x^{1/3}|(h,\ell)|}{D}\right) \ll \frac{DN}{x^{1/3}PK},$$

and the left-hand side of (6.12) is

$$\ll \frac{x^{1/6}}{P^{5/4}D^{1/2}K^{3/2}} \cdot PK^2M \cdot \frac{DN}{x^{1/3}PK} \ll x^{-1/6}D^{3/2} \ll x^{\theta_3}.$$

Turning to (6.13), we may remove the condition $mn \sim D$ from the sum to be estimated at the cost of a factor $\log x$, as noted earlier. Let us suppose this has been done. Let

$$Q = \max(64[x^{1/3}PKM^{-2}N^{-1}], 1).$$

We divide the interval $\left[0, \frac{8PH}{N}\right]$ into Q equal subintervals $I_1, \ldots, I_Q$, and bound

$$S = \sum_{m\sim M} a_m \sum_{n\sim N} c_n \sum_{(h,\ell)\in\mathcal{E}} b(h,\ell) e\left(\frac{-x^{1/3}|(h,\ell)|}{mn}\right)$$

as follows:

$$|S| \le \sum_{m\sim M} \sum_{q=1}^{Q} \left| \sum_{\substack{n\sim N,\,(h,\ell)\in\mathcal{E} \\ |(h,\ell)|/n\in I_q}} c_n b(h,\ell) e\left(-\frac{x^{1/3}|(h,\ell)|}{mn}\right) \right|.$$

Cauchy's inequality yields

(6.21)

$$|S|^2 \le MQ \sum_{q=1}^{Q} \sum_{m\sim M} \left| \sum_{\substack{n\sim N,\,(h,\ell)\in\mathcal{E} \\ |(h,\ell)|/n\in I_q}} c_n b(h,\ell) e\left(\frac{-x^{1/3}|(h,\ell)|}{mn}\right) \right|^2$$

$$\le MQ \sum_{\substack{n_1,n_2,(h_1,\ell_1),(h_2,\ell_2) \\ (6.22)}} \left| \sum_{m\sim M} e\left(-\frac{x^{1/3}}{m}\left(\frac{|(h_1,\ell_1)|}{n_1} - \frac{|(h_2,\ell_2)|}{n_2}\right)\right)\right|,$$

where n_1, n_2, (h_1,ℓ_1), (h_2,ℓ_2) are restricted in the last summation by

$$n_j \sim N, (h_j,\ell_j) \in \mathcal{E}\ ,\ \left|\frac{|(h_1,\ell_1)|}{n_1} - \frac{|(h_2,\ell_2)|}{n_2}\right| \le \frac{8PK}{NQ}. \tag{6.22}$$

A splitting-up argument yields

(6.23)

$$|S|^2 \ll MQ\log x \sum_{\substack{n_1,n_2,(h_1,\ell_1),(h_2,\ell_2) \\ (6.24)}} \left| \sum_{m\sim M} e\left(-\frac{x^{1/3}}{m}\left(\frac{|(h_1,\ell_1)|}{n_1} - \frac{|(h_2,\ell_2)|}{n_2}\right)\right)\right|,$$

where the outer summation (6.23) is restricted by $n_j \sim N$, $(h_j,\ell_j)\in\mathcal{E}$ and

$$\left(\Delta - \frac{MN}{x^{1/3}PK}\right)\frac{PK}{N} \le \left|\frac{|(h_1,\ell_1)|}{n_1} - \frac{|(h_2,\ell_2)|}{n_2}\right| < \frac{2\Delta PK}{N}. \tag{6.24}$$

The positive number Δ is of the form

$$\Delta = \frac{2^h MN}{x^{1/3}PK}\ ,\ \Delta \le \frac{8}{Q}\ ,\ h \ge 0.$$

We can apply the Kusmin-Landau theorem again, since

$$\left|\frac{d}{dm}\left(\frac{x^{1/3}}{m}\left(\frac{|(h_1,\ell_1)|}{n_1} - \frac{|(h_2,\ell_2)|}{n_2}\right)\right)\right| \le \frac{16x^{1/3}PK}{M^2NQ} \le \frac{1}{2}$$

by the choice of Q. Thus the inner sum in (6.23) is

$$\ll \min\left(M, \frac{M^2N}{x^{1/3}PK\Delta}\right).$$

According to Lemma 11, the number of solutions n_1, n_2, (h_1, ℓ_1), (h_2, ℓ_2) of (6.24) is

$$\ll x^{\epsilon}(\Delta P^3K^4N^2 + P^{3/2}K^3N).$$

Thus

$$\begin{aligned}|S|^2 &\ll x^{\epsilon}MQ\log x(\Delta P^3K^4N^2 + P^{3/2}K^3N)\min\left(M, \frac{M^2N}{x^{1/3}PK\Delta}\right)\\ &\ll x^{\epsilon}MQ\log x(P^2K^3ND^2x^{-1/3} + P^{3/2}K^3D),\\ S &\ll x^{\epsilon}Q^{1/2}PK^{3/2}(D^{3/2}x^{-1/6} + DN^{-1/2}).\end{aligned}$$

The left-hand side of (6.12) is now seen to be

$$\ll x^{1/6+\epsilon}P^{-1/4}D^{-1/2}Q^{1/2}(D^{3/2}x^{-1/6} + DN^{-1/2}).$$

To verify that this is $\ll x^{\theta_3+\epsilon}$ reduces to showing that

$$Q \ll \min(x^{2\theta_3}P^{1/2}D^{-2},\ x^{2\theta_3-1/3}P^{1/2}D^{-1}N). \tag{6.25}$$

If $Q = 1$, then (6.25) is a simple consequence of the lower bound $N \gg U$. Otherwise (6.25) reduces to the two assertions

$$x^{1/3}PKM^{-2}N^{-1} \ll x^{2\theta_3}P^{1/2}D^{-2} \tag{6.26}$$

and

$$x^{1/3}PKM^{-2}N^{-1} \ll x^{2\theta_3-1/3}P^{1/2}D^{-1}N. \tag{6.27}$$

Both assertions follow from (6.19). In the case of (6.26),

$$x^{1/3-2\theta_3}P^{1/2}KM^{-2}N^{-1}D^2 \ll x^{4/27-2\theta_3}D^{3/2} \ll 1.$$

In the case of (6.27),

$$x^{2/3-2\theta_3}P^{1/2}KM^{-2}N^{-2}D \ll x^{13/27-2\theta_3} \ll 1.$$

This completes the proof of (6.13). All the required bounds are now in place, and the proof of Theorem 1 is complete.

References

[1] T. M. Apostol, *Mathematical Analysis*, Addison Wesley 1957.

[2] R. C. Baker, *Sums of two relatively prime cubes*, Acta Arith. **129** (2007), 103–146.

[3] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.

[4] G. Harman, *Prime-detecting Sieves*, Princeton University Press, 2007.

[5] M. N. Huxley, *Exponential sums and the Riemann zeta function IV*, Proc. London Math. Soc. **66** (1993), 1–40.

[6] M. N. Huxley, *Area, lattice points and exponential sums*, Oxford University Press, 1996.

[7] M. N. Huxley, *Exponential sums and lattice points. III*, Proc. London Math. Soc. **87** (2003), 591–609.

[8] M. N. Huxley, *Exponential sums and the Riemann zeta function V*, Proc. London Math. Soc. (3) **90** (2005), 1–41.

[9] F. Jones, *Lebesgue integration on Euclidean space*, Jones and Bartlett 1993.

[10] G. Kuba, *On sums of two k-th powers in residue classes II*, Abh. Math. Sem. Hamburg **63** (1993) 87–95.

[11] M. Kühleitner, *On sums of two k-th powers: an asymptotic formula for the mean square of the error term*, Acta Arith. **92** (2000), 263–276.

[12] O. Robert and P. Sargos, *Three-dimensional exponential sums with monomials*, J. Reine Angew. Math. **591** (2006), 1–20.

[13] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1939.

[14] E. C. Titchmarsh, *The theory of the Riemann zeta function*, 2nd ed., Oxford University Press, 1986.

[15] J. D. Vaaler, *Some extremal problems in Fourier analysis*, Bull. Amer. Math. Soc. **12** (1985), 183–216.

[16] N. Watt, *Exponential sums and the Riemann zeta-function II*, J. London Math. Soc. **39** (1989), 385–404.

[17] J. Wu, *On the average number of unitary factors of finite abelian groups*, Acta Arith. **84** (1998), 17–29.

[18] W. Zhai, *On sums and differences of two coprime k-th powers*, Acta Arith. **91** (1999), 233–248.

[19] W. Zhai, *On sums of two coprime k-th powers*, Monatsh. Math. **144** (2005), 233–250.

[20] W. Zhai and X. Cao, *On sums of two coprime k-th powers (II)*, Monatsh. Math. **148** (2006), 153–172.

Department of Mathematics
Brigham Young University
Provo, UT 84602
U.S.A.

baker@math.byu.edu