# WEYL'S THEOREM IN THE MEASURE THEORY OF NUMBERS

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## 1. Introduction

We write $\{y\}$ for the fractional part of $y$. A real sequence $y_1, y_2, \ldots$ is said to be uniformly distributed $(\mathrm{mod}\, 1)$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{k\le N\\ \{y_k\}\in I}} 1 = m(I)$$

for each subinterval $I$ of $U = [0,1)$. Here $m(\ldots)$ denotes Lebesgue measure.

Let $\mathcal{S} = \{a_1 < a_2 < a_3 < \ldots\}$ be a strictly increasing sequence of real numbers with a spacing condition,

$$a_{k+1} - a_k \ge \sigma > 0 \quad (k = 1, 2, \ldots).$$

In his fundamental paper on uniform distribution [29], Hermann Weyl showed that the assertion

(1.1) $$a_1 x, a_2 x, a_3 x, \ldots \text{ is uniformly distributed } (\mathrm{mod}\, 1)$$

holds for almost all $x$ in $U$.

There are many ways of extending and refining this theorem about a sequence of dependent random variables. Here we review some of the literature, including results of Walter Philipp and his collaborators, and add some new theorems. A survey that complements the present article is given in chapter 5 of Harman [15], including improved proofs of some key results.

A result of particular interest is Salem's [26] strengthening of Weyl's assertion when

(1.2) $$a_k = O(k^p)$$

for a constant $p \ge 1$. When (1.2) holds, Salem showed that for a sequence of positive integers $\mathcal{S}$, (1.1) is valid except for a set of $x$ of Hausdorff dimension at most $1 - 1/p$. This result was also found by Erdös and Taylor [12]. The result was extended to real sequences $\mathcal{S}$ by Baker [3]. An example to show that the bound $1 - 1/p$ is attained for each $p$, with a sequence of positive integers $\mathcal{S}$, was given in Ruzsa [25].

*Date*: October 8, 2009.

A strengthening of Weyl's work that is valid for arbitrary $\mathcal{S}$ concerns the **discrepancy** of the sequence (1.1). For a subinterval $I$ of $U$, let

$$Z(N,I,x) = |\{k \le N : \{a_k x\} \in I\}|. \tag{1.3}$$

Here $|E|$ denotes the cardinality of a finite set $E$. Let

$$D(N,x) = \sup_{I\subset U} |Z(N,I,x) - Nm(I)|, \tag{1.4}$$

where the supremum is taken over all subintervals of $U$.

The definition of uniform distribution (mod 1) is equivalent to

$$D(N,x) = o(N) \text{ as } N \to \infty. \tag{1.5}$$

It is known that

$$D(N,x) = O(N^{1/2}(\log N)^{3/2+\epsilon}) \text{ a.e.} \tag{1.6}$$

(Baker [4] for integer sequences; the general case is given by Harman [15]). An example of Berkes and Philipp [6] shows that the constant 3/2 cannot be reduced below 1/2.

For a **lacunary** sequence, namely a sequence $\mathcal{S}$ with

$$\frac{a_{j+1}}{a_j} \ge c > 1 \qquad (j = 1, 2, \ldots)$$

we can be more precise:

$$\frac{1}{4} \le \limsup_N \frac{D(N,x)}{\sqrt{N \log\log N}} \le f(c) \text{ a.e.}$$

This version of the law of the iterated logarithm is due to Philipp [22]. See Berkes, Philipp, and Tichy [7] for further results of this kind; also the papers in the present volume by Aistleitner and Fukuyama.

One way of extending Weyl's theorem is to interpret $x$ in (1.1) as a point of $\mathbb{R}^d$ and $\{a_j x\}$ as the unique point $a_j x - k, k \in \mathbb{Z}^d$, that lies in $U^d$. The definition (1.4) must be modified; $U$ is replaced by $U^d$, and the symbol $I$ now denotes a box, that is, a Cartesian product of subintervals of $U$. Again, the definition of uniform distribution (mod 1) in [29] and subsequent work is equivalent to (1.5).

The extension of versions of (1.5) to this situation is discussed in [15, 21]. However, it seems that the following extension of Salem's theorem is new. For brevity write $E^d(\mathcal{S})$ for the set of $x$ in $U^d$ for which the sequence $a_1x,\ a_2x,\ \ldots$ is **not** uniformly distributed (mod 1).

**Theorem 1.** *Suppose that* (1.2) *holds. Then*

$$\dim E^d(\mathcal{S}) \ \le\ d - \frac{1}{p}.$$

Without much effort, we can deduce the following from Rusza's work.

**Theorem 2.** *For every $p \ge 1$, there exists a strictly increasing sequence of positive integers $\mathcal{S}$ satisfying* (1.2), *for which*

$$\dim E^d(\mathcal{S}) \ =\ d - \frac{1}{p}.$$

In metric diophantine approximation, a lot of effort goes into discovering what happens at almost all points on a curve $\mathcal{C}$ in $\mathbb{R}^d$. See for example Kleinbock and Margulis [20]. For a sharp planar result and references to the recent literature, see Vaughan and Velani [28]. However, it seems not to have been asked whether the intersection of $E^d(\mathcal{S})$ with a suitable curve is a null subset of the curve.

Let $\mathcal{C}$ be a curve given by

$$x = x(t) = (x_1(t), \ldots, x_d(t)) \quad (a \le t \le b) \tag{1.7}$$

where ${x_j}'$ is continuous ($1 \le j \le d$). For a sequence $\mathcal{S}$ of integers, a necessary condition for a result of the type mentioned is that the functions $1, x_1, \ldots, x_d$ are linearly independent over the rationals. In the contrary case, we have a relation

$$h_1 x_1(t) + \ldots + h_d x_d(t) = h_{d+1} \quad (a \le t \le b)$$

with integers $h_j$ not all 0. The point $k$ with $a_j x - k = \{a_j x\}$ satisfies

$$h_1(a_j x_1(t) - k_1) + \ldots + h_d(a_j x_d(t) - k_) \in \mathbb{Z}$$

or

$$h\{a_j x(t)\} = \alpha$$

where there are only finitely many possibilities for the integer $\alpha$ as $t$ and $j$ vary. (We write $hy$ for the inner product if $h \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$). This restricts $\{a_j x\}$ to points on a finite number of hyperplanes that intersect $U^d$, and precludes uniform distribution.

The following positive result restricts $\mathcal{C}$ in a reasonable way, although it would be nice to require the existence of fewer derivatives.

**Theorem 3.** *Suppose that $x(t)$, given by* (1.6)*, satisfies*

(1) ${x_j}^{(d+1)}(t)$ *exists and is bounded* $(1 \le j \le d)$*;*

(2) *The matrix*

$$A(t) = [{x_i}^{(j)}(t)] \quad (1 \le i, j \le d)$$

*is non-singular* $(a \le t \le b)$.

*Then* (1.1) *holds with $x = x(t)$, except for a null set of $t$.*

Once Theorem 3 is proved, it is easy to relax (2) to the assertion '$A(t)$ is non-singular a.e.' This is left as an exercise for the interested reader.

An imperfect analogue of Salem's theorem is:

**Theorem 4.** *Make the hypotheses of Theorem 3. Suppose further that* (1.2) *holds. Then the set*

$$\{t \in [a,b] : (1.1) \text{ fails for } x = x(t)\}$$

*has Hausdorff dimension at most* $1 - \frac{1}{pd}$.

For the remainder of this section, let $d = 1$, and suppose that $\mathcal{S}$ is a sequence of positive integers. We examine particularly bad failures of the assertion (1.1). We say that the sequence $a_1 x, a_2 x, \ldots$ is **almost uniformly distributed** (mod 1) if there is a sequence $M_k \to \infty$ such that

$$M_k^{-1}\, Z(M_k, I, x) \to m(I)$$

for all subintervals $I$ of $U$. Let us write

$$F(\mathcal{S}) = \{x \in U : a_1x, a_2x, \ldots \text{ is } \textbf{not} \text{ almost uniformly distributed } (\mathrm{mod}\, 1)\},$$

so that

$$F(\mathcal{S}) \subseteq E^1(\mathcal{S})$$

Piatetskii-Sapiro [23] showed that for subsequences $\mathcal{S}$ with

$$a_k = O(k), \tag{1.8}$$

the set $F(\mathcal{S})$ is **countable**. This may be surprising at first. Baker [2] constructed a sequence with

$$1 \le a_{k+1} - a_k \le 2 \quad (k \ge 1)$$

for which $E^1(\mathcal{S})$ is uncountable. (This is a slight strengthening of a result in [12].)

If the sequence $a_1x, a_2x, \ldots$ is almost uniformly distributed, then obviously

$$\limsup_N \frac{Z(N, I, x)}{N} \ge m(I) \tag{1.9}$$

for every subinterval $I$ of $U$; there is, of course, a corresponding statement about the lim inf. We say that the sequence $a_1x, a_2x, \ldots$ is **biased** if (1.9) fails for some interval $I$. The **bias** of the sequence is then

$$b(x) = \sup_{I \subset U} \quad \{m(I) - \limsup_N \frac{Z(N, I, x)}{N}\}$$

Let $B(\mathcal{S})$ be the set of $x$ in $U$ for which $b(x) > 0$. By the above remarks,

$$B(\mathcal{S}) \subseteq F(\mathcal{S}).$$

Kahane [16], unaware of [23], showed that (1.8) implies the countability of $B(\mathcal{S})$. He deduced this from the following finiteness result, which does not emerge from the method of [23].

**Theorem 5.** *Let $\mathcal{S}$ be a strictly increasing sequence of positive integers. Let $C > 0, \delta > 0$. Let $I$ be a subinterval of $U$. Suppose that*

$$a_k \le Ck \quad \textit{for infinitely many } k. \tag{1.10}$$

*The set of $x$ for which*

$$\frac{Z(N, I, x)}{N} \le m(I) - \delta \textit{ for } N \ge 1$$

*is finite.*

In particular, the set of $x$, say $H_I(\mathcal{S})$, for which

$$a_k x \notin I \,(\mathrm{mod}\, 1) \qquad (k \ge 1) \tag{1.11}$$

is finite under the hypothesis (1.8). This result was found independently by Amice [1]. There is an interesting variant due to Kaufman [19]. Let $I$ be a box in $U^d$. if each of the sequences $\mathcal{S}_1, \ldots, \mathcal{S}_d$ satisfies condition (1.10), then the set of $x$ in $U$ for which

$$x(a_1, \ldots, a_k) \notin I \,(\mathrm{mod}\, 1) \qquad (a_j \in \mathcal{S}_j, a_1 < \ldots < a_d)$$

is **countable**.

In this connection, we mention Boshernitzan's result [8] that $H_I(\mathcal{S})$ has Hausdorff dimension 0 under the condition

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$$

For a lacunary sequence $\mathcal{S}$, quite the opposite is true. There is a subinterval $I$ of $U$ for which $H_I(\mathcal{S})$ has Hausdorff dimension 1. This result was found independently by de Mathan [10, 11] and Pollington [24].

The following strengthening of Theorem 5 seems to have been overlooked.

**Theorem 6.** *Let $\mathcal{S}$ be a strictly increasing sequence of positive integers. Let $C$ be a positive integer and $0 < \delta < 1$. Assume that* (1.10) *holds.*

*(i) The set $B_\delta(\mathcal{S})$ of $x$ in $U$ for which $b(x) \ge \delta$ is finite. In fact*

$$|B_\delta(\mathcal{S})| \le 144\, C \left(\log\left(\frac{2e}{\delta}\right)\right)^2 \delta^{-3}.$$

*(ii) Let $I$ be a subinterval of $U$. Then*

$$|H_I(\mathcal{S})| \le \min\left(\frac{288\, C}{m(I)^3}, \frac{144(C\log\left(\frac{2e}{m(I)}\right))^2}{m(I)^2}\right).$$

Part (ii) is not far from the truth for $m(I)$ small. Let $C$ be a positive integer and $0 < \delta < 1/2$. Let $E$ be the set of rational numbers in $U$ of the form

$$\frac{r}{sC}, \qquad 0 \le r < sC, \quad (r,s) = 1, \quad s \le \frac{1}{\delta}.$$

Clearly

$$|E| \gg \frac{C}{\delta^2}$$

in view of the average order of the $\phi$-function; see Hardy and Wright [14, Theorem 330]. Let $a_j = Cj$. Then for $x \in E, x = \frac{r}{sC}$, we have

$$\{a_j x\} = \{\frac{jr}{s}\} \notin I := (0, \delta).$$

Thus $E \subseteq H_I(\mathcal{S})$, and

$$|H_I(\mathcal{S})| \gg \frac{C}{m(I)^2}.$$

We conjecture that, in general, Theorem 6 (ii) could be improved to

$$|H_I(\mathcal{S})| \ll_\epsilon \left(\frac{C}{m(I)^2}\right)^{1+\epsilon}$$

for every $\epsilon > 0$.

## 2. Proofs of Theorems 1 and 2

Let $||\dots||$ denote Euclidean length. We write $D(X) = \sup\{||x-y|| : x, y \in X\}$ for $X \subset \mathbb{R}^d$.

**Lemma 1.** *Let $F$ be a non-negative function on*

$$J = [a_1, b_1] \times \cdots \times [a_d, b_d].$$

*Suppose that $\frac{\partial F}{\partial x_i}$ exists, $\frac{\partial F}{\partial x_i} \le A \quad (1 \le i \le d, x \in J)$, and*

$$\int_J F(x) dx \le B.$$

*Let $0 < c < 2Ad \min_j (b_j - a_j)$. Define*

$$E = \{x \in J : F(x) \ge c\}.$$

*There is a covering of $E$ with boxes $I_1, \dots, I_q$ such that, for $0 < \gamma < d$,*

$$\sum_{j=1}^{q} D(I_j)^\gamma \ll_d BA^{d-\gamma} c^{\gamma-(d+1)}. \tag{2.1}$$

*Proof.* Let

$$M_j = \left[\frac{2Ad}{c}(b_j - a_j)\right] + 1 \le \frac{4Ad}{c}(b_j - a_j).$$

We partition $J$ into $M_1 \dots M_d$ boxes whose sides have respective lengths $(b_j - a_j)/M_j \quad (1 \le j \le d)$. Note that

$$\frac{c}{4Ad} \le \frac{b_j - a_j}{M_j} \le \frac{c}{2Ad}.$$

Among these $M_1 \dots M_d$ boxes, suppose that $I_1, \dots, I_q$ are those that meet $E$. Now

$$F(x) \ge \frac{c}{2} \qquad \text{on } I_l$$

by applying the mean value theorem $d$ times. Hence

$$\frac{c}{2} q \prod_{j=1}^{d} \frac{b_j - a_j}{M_j} = \frac{c}{2} \sum_{l=1}^{q} m(I_l) \le B.$$

So

$$q \le \frac{2B}{c} \prod_{j=1}^{d} \frac{M_j}{b_j - a_j} \le \frac{2B}{c} \left(\frac{4Ad}{c}\right)^d.$$

By Hölder's inequality,

$$\begin{aligned} \sum_{l=1}^{q} D(I_l)^\gamma &\le q^{1-\gamma/d} \left(\sum_{l=1}^{q} D(I_l)^d\right)^{\gamma/d} \\ &\ll_d q^{1-\gamma/d} \left(\sum_{l=1}^{q} m(I_l)\right)^{\gamma/d} \\ &\ll_d \left(\frac{BA^d}{c^{d+1}}\right)^{1-\gamma/d} \left(\frac{B}{c}\right)^{\gamma/d} \end{aligned}$$

as required. ∎

2.1. **Proof of Theorem 1.**

Naturally we may confine attention to $x$ in a fixed box $[-K,K]^d$. Let $h = (h_1,\dots,h_d)$ be any nonzero point of $\mathbb{Z}^d$. By Weyl's criterion, it suffices to show that the set

$$Z = \{x \in [-K,K]^d : \left|\sum_{k=1}^{N} e(a_k hx)\right| > N(\log N)^{-\frac{1}{2}} \quad \text{for infinitely many } N\}$$

has dimension at most $d - \frac{1}{p}$. Here $e(\theta)$ denotes $e^{2\pi i\theta}$.

Let

$$N_r = [e(r^{\frac{1}{2}})].$$

Then

$$\frac{N_{r+1}}{N_r} - 1 \le \frac{\exp((r+1)^{\frac{1}{2}})}{\exp(r^{\frac{1}{2}}) - 1} - 1 \ll r^{-\frac{1}{2}}.$$

Suppose $N$ is a large positive integer with

$$\left|\sum_{k=1}^{N} e(a_k hx)\right| > N(\log N)^{-\frac{1}{2}},$$

say

$$N_r \le N < N_{r+1}.$$

Then

$$\begin{aligned}
\left|\sum_{k\le N_r} e(a_k hx)\right| &> \frac{N}{(\log N)^{\frac{1}{2}}} - (N - N_r) \\
&> \frac{N_r}{(\log N_r)^{\frac{1}{2}}} - (N_{r+1} - N_r) \\
&= N_r\left(\frac{1}{(\log N_r)^{\frac{1}{2}}} + \mathrm{O}(r^{-\frac{1}{2}})\right) \\
&> \frac{N_r}{2(\log N_r)^{\frac{1}{2}}}
\end{aligned}$$

since $(\log N_r)^{-\frac{1}{2}} \ge r^{-\frac{1}{4}}$. Hence

$$\begin{aligned}
Z &\subseteq \{x \in [-K,K]^d : \left|\sum_{k=1}^{N_r} e(a_k hx)\right| > \frac{N_r}{2(\log N_r)^{\frac{1}{2}}} \quad \text{for infinitely many } r\} \\
&= \bigcap_{m=1}^{\infty} \bigcup_{r=m}^{\infty} E_r.
\end{aligned}$$

Here

$$E_r = \left\{x \in [-K,K]^d : \left|\sum_{k=1}^{N_r} e(a_k hx)\right| > \frac{N_r}{2(\log N_r)^{\frac{1}{2}}}\right\}.$$

We now apply Lemma 1 with $J = [-K, K]^d$,

$$F(x) = \left|\sum_{k=1}^{N_r} e(a_k hx)\right|^2$$

$$= \sum_{k=1}^{N_r}\sum_{l=1}^{N_r} e((a_k - a_l)hx),$$

$$\frac{\partial F}{\partial x_j} = \sum_{k=1}^{N_r}\sum_{l=1}^{N_r} 2\pi i(a_k - a_l)h_j e((a_k - a_l)hx)$$

$$\leq C_1 N_r^{p+2},$$

where $C_1 = C_1(h, \mathcal{S}) \geq 1$. Take $A = C_1 N_r^{p+2}$, $c = N_r^2(4\log N_r)^{-1}$, so that $c < 2Ad$ as required. With $C_2 = C_2(\sigma, K)$, we have

$$\int F(x)\, dx \;\leq\; C_2 N_r \log N_r \;=\; B.$$

To see this, suppose for example that $h_1 \neq 0$. Then

$$\left|\int_{[-K,K]^{d-1}} dx_2 \ldots dx_d \int_{-K}^{K} F(x)\, dx_1\right| \leq (2K)^{d-1}\sum_{k=1}^{N_r}\sum_{l=1}^{N_r}\left|\int_{-K}^{K} e((a_k - a_l)h_1 x_1)\, dx_1\right|$$

$$\leq (2K)^d N_r + 2\sum_{k=1}^{N_r}\sum_{l=1}^{k-1}\frac{1}{a_k - a_l}$$

$$\leq (2K)^d N_r + \frac{2}{\sigma}\sum_{k=1}^{N_r}\sum_{l=1}^{k-1}\frac{1}{k-l}$$

$$< C_2 N_r \log N_r.$$

Finally, let $0 < \epsilon < \frac{1}{2p}$ and take

$$\gamma \;=\; d - \frac{1}{p} + 2\epsilon.$$

By Lemma 1, we can cover $E_r$ with boxes $I_{r1}, \ldots, I_{rq}$, $q = q(r)$ such that

$$\sum_{l=1}^{q} D(I_{r\ell})^\gamma \;\ll_{d,\mathcal{S},h}\; N_r \log N_r (N_r^{p+2})^{d-\gamma}\left\{\frac{N_r^2}{\log N_r}\right\}^{\gamma-(d+1)}.$$

Hence

$$\sum_{l=1}^{q} D(I_{r\ell})^\gamma \;<\; N_r^{pd-1-p\gamma+\epsilon}$$

for large $r$. The exponent of $N_r$ here is negative and so

$$\sum_{r=1}^{\infty}\sum_{l=1}^{q(r)} D(I_{r\ell})^\gamma \;<\; \infty.$$

By choosing $m$ large, we cover $\bigcup_{r=m}^{\infty} E_r$ with boxes $\{I_{r\ell} : r \geq m,\ 1 \leq l \leq q(r)\}$ for which the sum

$$\sum_{r=1}^{\infty} \sum_{l=1}^{q(r)} D(I_{r\ell})^{\gamma}$$

is arbitrarily small. These intervals cover $Z$, and so $Z$ has dimension at most $\gamma$. The theorem follows at once. □

2.2. **Proof of Theorem 2.**

For any non-empty subsets $A,\ B$ of $\mathbb{R}$,

$$\dim(A \times B) \geq \dim A + \dim B$$

(Falconer [13, Corollary 5.10]).

We may suppose that $d \geq 2$. Now

$$E^d(\mathcal{S}) \supseteq E^1(\mathcal{S}) \times U^{d-1}.$$

For suppose that $(x_1, \ldots, x_d) \notin E^d(\mathcal{S})$. We claim that $x_1 \notin E^1(\mathcal{S})$. Indeed for $I \subseteq U$,

$$\frac{1}{N} Z(N, I, x_1) \;=\; \frac{1}{N} Z(N, I \times U^{d-1}, (x_1, \ldots, x_d)) \to m_d(I \times U^{d-1}) = m_1(I)$$

as $N \to \infty$.

Now let $p \geq 1$ and let $\mathcal{S}$ be the sequence satisfying (1.2) for which

$$\dim E^1(\mathcal{S}) = 1 - \frac{1}{p}$$

constructed by Ruzsa [25]. (We may choose any $\mathcal{S}$ with $a_k = \mathrm{O}(k)$ for $p = 1$.) We have

$$\begin{aligned} \dim E^d(\mathcal{S}) &\geq \dim(E^1(\mathcal{S}) \times U^{d-1}) \\ &\geq \dim E^1(\mathcal{S}) + \dim U^{d-1} \\ &= 1 - \frac{1}{p} + d - 1 = d - \frac{1}{p}. \end{aligned}$$

Since we know already that $\dim E^d(\mathcal{S}) \leq d - \frac{1}{p}$, Theorem 2 follows. □

It is interesting to observe that in Lemma 1, the exponents attached to $B, A, c$ cannot be improved in the following sense. If there are constants $e_1, \ldots e_5$ such that the left-hand side of (2.1) is always

$$\ll_d B^{e_1} A^{e_2 - e_3 \gamma} c^{e_4 \gamma - e_5}$$

the we cannot have $e_1 \leq 1$, $e_2 \leq d$, $e_3 \geq 1$, $e_4 \leq 1$, $e_5 \geq d+1$ unless equality holds in all five cases. Otherwise we could clearly obtain a better bound than $d - \frac{1}{p}$ for $\dim E^d(\mathcal{S})$, in contradiction to Theorem 2.

## 3. Proof of Theorems 3 and 4.

**Lemma 2.** *Let $\epsilon > 0$. Let $f$ be a real function on $[a,b]$. Suppose that $f^{(m)}$ is continuous and*

$$|f^{(m)}(t)| \geq 1 \qquad (a \leq t \leq b)$$

*After excluding $2^m - 1$ pairwise disjoint intervals of length $\leq 2\epsilon^{\frac{1}{m}}$ from $[a,b]$ we have*

$$|f(t)| \geq \epsilon. \tag{3.1}$$

*Proof.* By making a sign change if necessary, we may replace the hypothesis by

$$f^{(m)}(t) \geq 1 \qquad (a \leq t \leq b). \tag{3.2}$$

We prove the assertion (for all $f$, $\epsilon$) using induction on $m$.

For $m = 1$, $f$ is strictly increasing, and

$$\{t \in [a,b] : -\epsilon < f(t) < \epsilon\}$$

is an interval $I$ (possibly empty). Say $\overline{I} = [u,v]$. The mean value theorem yields

$$\min_{x \in [a,b]} f'(x)(v-u) \leq f(v) - f(u) \leq 2\epsilon$$

giving $v - u \leq 2\epsilon$ as required.

Suppose the assertion has been proved for all $f$, $\epsilon$, with $1, \ldots, m-1$ in place of $m$. Let $\eta = \epsilon^{1/m}$. By the case $m = 1$, (3.2) implies

$$\left|f^{(m-1)}(t)\right| \geq \eta$$

after excluding an interval $I_0$ of length $\leq 2\eta$. Let $I$ be one of the intervals complementary to $I_0$ in $[a,b]$. Then $g = \eta^{-1} f$ has

$$\left|g^{(m-1)}(t)\right| \geq 1 \qquad \text{on } \overline{I}.$$

By the case $m-1$, after excluding $2^{m-1} - 1$ pairwise disjoint intervals in $\overline{I}$, each of length $\leq 2(\frac{\epsilon}{\eta})^{\frac{1}{m-1}} = 2\epsilon^{1/m}$ we have

$$|g(t)| \geq \frac{\epsilon}{\eta},$$

that is, $|f(t)| \geq \epsilon$.

Thus after excluding $I_0$ and $2(2^{m-1} - 1) = 2^m - 2$ other intervals of length $\leq 2\epsilon^{1/m}$ in $[a,b] \backslash I_0$, the whole family of $2^m - 1$ intervals having pairwise disjoint interiors, we have (3.1). After adjusting the endpoints of abutting intervals, this completes the induction step and proves Lemma 1. □

By considering the example $f(t) = \frac{t^m}{m!}$, it is easy to see that the lemma is sharp for given $m$ apart from the value of the constant $2^m - 1$.

**Lemma 3.** *Let $0 < \epsilon < 1$, $m \geq 2$. Suppose that $f$ is a real function on $[a, b]$ with bounded $(m+1)$-th derivative. Let*

$$C = \max\left\{\left|f^{(j)}(t)\right| : \quad a \leq t \leq b,\ 2 \leq j \leq m+1\right\}.$$

*Suppose further that*

$$\max\left\{|f'(t)|, \ldots, |f^{(m)}(t)|\right\} \geq B \qquad \text{on } [a, b].$$

*After excluding at most*

$$\left(\frac{C(b-a)}{B} + 1\right)(2^{m-1} - 1)$$

*pairwise disjoint intervals of length at most $2\epsilon^{\frac{1}{m-1}}$ from $[a, b]$ we have*

$$|f'(t)| \geq \frac{B\epsilon}{2}. \tag{3.3}$$

*Proof.* We divide $[a, b]$ into $\left[\frac{C(b-a)}{B}\right] + 1$ pairwise disjoint intervals $I_1$, $I_2$, $\ldots$ of length $\leq \frac{B}{C}$. At the midpoint of $I_k$, there is a $j = j_k$, $1 \leq j \leq m$, with

$$\left|f^{(j)}(t)\right| \geq B.$$

By the mean value theorem, $|f^{(j)}(t)| \geq B/2$ in $I_k$. If $j \geq 2$, we find that

$$\left|\frac{2f'}{B}\right| \geq \epsilon$$

on $I_k$ after excluding at most $2^{m-1} - 1$ intervals of length $2\epsilon^{1/(j-1)} \leq 2\epsilon^{1/(m-1)}$; this is an application of Lemma 2 with $\frac{2f'}{B}$, $j-1$ in place of $f$, $m$. The lemma follows on summing the total number of excluded intervals contained in those $I_k$ with $j_k \geq 2$. □

In the remainder of this section, $C_1, C_2, \ldots$ denote positive constants depending only on $h$, $\mathcal{S}$ and on the function $x(.) = (x_1(.), \ldots, x_d(.))$

**Lemma 4.** *Make the hypotheses of Theorem 3. Let $h \in \mathbb{Z}^d$, $h \neq 0$,*

$$f(t) = hx(t). \tag{3.4}$$

*Then*

$$\max\left\{|f'(t)|, \ldots, |f^{(d)}(t)|\right\} \geq C_1 \qquad \text{on } [a, b].$$

*Proof.* Fix $t \in [a, b]$. We have

$$\begin{bmatrix} f'(t) \\ \vdots \\ f^{(d)}(t) \end{bmatrix} = A(t) \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix};$$

so that, writing

$$A(t)^{-1} = [c_{ij}(t)]$$

$$\begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} = A(t)^{-1} \begin{bmatrix} f'(t) \\ \vdots \\ f^{(d)}(t) \end{bmatrix},$$

we have

$$1 \le \max(|h_1|, \dots, |h_d|) \le \max_{j \le d} (|c_{j1}(t)| + \cdots + |c_{jd}(t)|) \max \left| f^{(j)}(t) \right|$$
$$\le C_2 \max_{j \le d} \left| f^{(j)}(t) \right|,$$

since the determinant $\det A(t)$ is bounded away from zero. □

**Lemma 5.** *Make the hypotheses of Lemma 4. Let*

$$F_N(t) = \left| \sum_{k=1}^{N} \exp\left(a_k h x(t)\right) \right|^2,$$

*then*

$$\int_a^b F_N(t)\, dt \le C_3 N^{2-\frac{1}{d}} (\log N)^{1/d}.$$

*Proof.* Define $f(t)$ by (3.4). Let

$$\lambda = \left( \frac{\log N}{N} \right)^{\frac{d-1}{d}}.$$

By Lemmas 3 and 4, we may partition $[a, b]$ into intervals $I_1, \dots I_l, J_1, \dots, J_k$ with $l \le k + 1 \le C_4$ and

$$|f'(t)| \ge C_5 \lambda \qquad (t \in \bigcup_{i \le l} I_i),$$

$$m(J_i) \le 2\lambda^{\frac{1}{(d-1)}} \qquad (1 \le i \le k).$$

Trivially,

$$\int_{J_i} F_N(t)\, dt \le N^2 m(J_i) \le 2(\log N)^{1/d} N^{2-\frac{1}{d}}.$$

Now

$$\int_{I_i} F_N(t)\, dt \;=\; N m(I_i) + 2 \ \mathrm{Re} \sum_{k=1}^{N} \sum_{1 \le j < k} \int_{I_i} \exp((a_k - a_j) f(t))\, dt$$
$$\le N m(I_i) + 8 \sum_{k=1}^{N} \sum_{1 \le j < k} \frac{1}{a_k - a_j} \max_{t \in I_i} \frac{1}{|f'(t)|}$$

(by a standard lemma; see [27, Lemma 4.2]). Thus

$$\int_{I_i} F_N(t)\, dt < C_6 \left( N + \frac{1}{\sigma\lambda} \sum_{k=1}^{N} \sum_{1 \le j < k} \frac{1}{k - j} \right)$$
$$< C_7 \frac{N \log N}{\lambda} = C_7 N^{2-\frac{1}{d}} (\log N)^{1/d}.$$

The lemma follows on assembling these upper bounds. □

3.1. **Proof of Theorem 3.** By Weyl's criterion, we need only show for fixed $h \in \mathbb{Z}^k,\ h \neq 0$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \exp(hx(t)) = 0 \tag{3.5}$$

for almost all $t$. Let $F_N(t)$ be as in Lemma 5. According to (a slight variant of) a theorem of Davenport, Erdös and LeVeque [9], a sufficient condition for (3.5) to hold a.e. is

$$\sum_{N=1}^{\infty} N^{-3} \int_a^b F_N(t)\,dt < \infty$$

We complete the proof on an application of Lemma 5. □

3.2. **Proof of Theorem 4.** Let

$$\gamma = 1 - \frac{1}{pd} + 2\epsilon,$$

where $0 < \epsilon < \frac{1}{2pd}$. As in the proof of Theorem 1, it suffices to show that

$$Z^{(1)} = \left\{ t \in [a,b] : \left| \sum_{k=1}^{N} \exp(a_k hx(t)) \right| > N(\log N)^{-1/2} \text{ for infinitely many } N \right\}$$

has dimension at most $\gamma$. Here $h$ is a fixed nonzero element of $\mathbb{Z}^d$.

Just as in that proof,

$$Z^{(1)} \subseteq \bigcap_{m=1}^{\infty} \bigcup_{r=m}^{\infty} E_r^{(1)},$$

where

$$E_r^{(1)} = \left\{ t \in [a,b] : \left| \sum_{k=1}^{N_r} \exp(a_k hx(t)) \right| > \frac{N_r}{(2\log N_r)^{1/2}} \right\}$$

We apply Lemma 1 with 1 in place of $d$, and

$$F(t) = \left| \sum_{k=1}^{N_r} \exp(a_k hx(t)) \right|^2,$$

$$F'(t) = 2\pi i \sum_{k=1}^{N_r} \sum_{l=1}^{N_r} (a_k - a_l) hx'(t) \exp((a_k - a_l)hx(t))$$

$$\leq C_8 N_r^{p+2}.$$

Thus in Lemma 1 we take

$$A = C_8 N_r^{p+2}, \qquad B = C_9 N_r^{2-1/d} (\log N_r)^{1/d}$$

(recalling Lemma 5) and

$$c = \frac{N_r^2}{4 \log N_r}.$$

We can cover $E_r^{(1)}$ with intervals $I_{r1}, \ldots, I_{rq}$, $q = q(r)$, such that

$$\sum_{l=1}^{q} |I_{rl}|^\gamma \ll C_9 N_r^{2-\frac{1}{d}} (\log N_r)^{1/d} (C_8 N_r^{p+2})^{1-\gamma} \left( \frac{N_r^2}{4 \log N_r} \right)^{\gamma - 2}.$$

Hence

$$\sum_{l=1}^{q} |I_{rl}|^\gamma < N_r^{p-1/d-p\gamma+\epsilon}$$

for large $r$. The exponent of $N_r$ is negative. Just as in the proof of Theorem 1, $\dim Z^{(1)} \leq \gamma$, and the theorem follows. □

## 4. Proof of Theorem 6

This is rather similar to Kahane's argument. That argument is in turn adapted from Kahane and Salem [17]. In [17], $\mathcal{S}$ is arbitrary, and $B(\mathcal{S})$ is shown not to support a positive Borel measure with Fourier-Stieltjes coefficients vanishing at infinity. We require two standard lemmas. For a subinterval $I$ of $U$, write

$$\Phi_I(x) = \begin{cases} 1 & \text{if } \{x\} \in I \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 6.** *Let $L$ be a natural number. For any subinterval $I = [a, b)$ of $U$, there is a trigonometric polynomial*

$$T(x) = \sum_{l=-L}^{L} c_l \exp(lx)$$

*satisfying*

$$T(x) \leq \Phi_I(x), \tag{4.1}$$

$$c_0 = m(I) - \frac{1}{L+1}, \tag{4.2}$$

$$|c_l| \leq \min\left( \frac{3}{2|l|}, \frac{1}{L+1} + m(I) \right) \qquad (l \neq 0). \tag{4.3}$$

*Proof.* This is obtained by combining Lemma 2.7 and (2.20) of [5], supplemented by the inequality $|\sin \alpha| \leq |\alpha|$. □

**Lemma 7.** *Let $x_1, \ldots, x_u$ be distinct points of $U$. Then*

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{s=-N}^{N} \left| \sum_{t=1}^{u} b_t \exp(s x_t) \right|^2 = \sum_{t=1}^{u} |b_t|^2.$$

*Proof.* Let $\mu$ be the measure on $U$ given by

$$\mu(E) = \sum_{x_t \in E} \overline{b_t}.$$

According to Wiener ([18], p. 42, Corollary),

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{s=-N}^{N} |\widehat{\mu}(s)|^2 = \sum_{\tau} |\mu(\{\tau\})|^2.$$

The lemma follows at once. □

4.1. **Proof of Theorem 6.** Let $r$ be a natural number and $0 < \beta < 1$. We first obtain an upper bound for the cardinality of the set $H(r, \beta)$ of $x$ in $U$ for which there exists a subinterval $I$ of $U$ with

$$\frac{Z(N, I, x)}{N} \leq m(I) - \beta \qquad (N \geq r).$$

We may suppose $H(r, \beta)$ is nonempty. Let $x_1, \ldots, x_u$ be distinct points of $H(r, \beta)$. For $t = 1, \ldots, u$ let $I_t$ be an interval such that

$$\sum_{k=1}^{N} \Phi_{I_t}(a_k x_t) \leq N(m(I_t) - \beta) \quad (N \geq r)$$

Let $T_t(x)$ be the trigonometric polynomial in Lemma 6 with $I = I_t$, $\quad L = \left[\frac{2}{\beta}\right]$,

$$T_t(x) = \sum_{l=-L}^{L} c_t(l) \exp(lx).$$

then for given $t$ and $N \geq r$,

$$N\left(m(I_t) - \frac{1}{L+1}\right) + \sum_{k=1}^{N} \sum_{0<|l|\leq L} c_t(l) \exp(l a_k x_t)$$
$$= \sum_{k=1}^{N} T_t(a_k x_j) \leq N(m(I_t) - \beta).$$

Since $\frac{1}{L+1} < \frac{\beta}{2}$,

$$\sum_{k=1}^{N} \sum_{0<|l|\leq L} c_t(l) \exp(l a_k x_t) \leq -\frac{N\beta}{2},$$

and summing over $t$,

$$\sum_{0<|l|\leq L} \sum_{k=1}^{N} \sum_{t=1}^{u} c_t(l) \exp(l a_k x_t) \leq -\frac{Nu\beta}{2} \tag{4.4}$$

for $N \geq r$; so that

$$\sum_{0<|l|\leq L} \left| \sum_{k=1}^{N} \sum_{t=1}^{u} c_t(l) \exp(l a_k x_t) \right| \geq \frac{Nu\beta}{2}. \tag{4.5}$$

For $l$ counted in (4.5), Cauchy's inequality gives

$$\left| \sum_{k=1}^{N} \sum_{t=1}^{u} c_t(l) \exp(l a_k x_t) \right|^2 \leq N \sum_{k=1}^{N} \left| \sum_{t=1}^{u} c_t(l) \exp(l a_k x_t) \right|^2 \tag{4.6}$$
$$\leq N \sum_{s=-CLN}^{CLN} \left| \sum_{t=1}^{u} c_t(l) \exp(s x_t) \right|^2$$

whenever $N$ satisfies $n_N \le CN$. The last expression in (4.6) is

$$\le (2+\epsilon)CLN^2 \sum_{t=1}^{u} |c_t(l)|^2$$

by Lemma 7 if, in addition, $N$ is sufficiently large. Comparing this with (4.5), we find that

$$\frac{u\beta}{2} \le \sum_{0<|l|\le L} (2CL)^{1/2} \left( \sum_{t=1}^{u} |c_t(l)|^2 \right)^{1/2}, \tag{4.7}$$

since $\epsilon$ is arbitrary.

Recalling (4.3),

$$\left( \sum_{t=1}^{u} |c_t(l)|^2 \right)^{1/2} \le \frac{3u^{1/2}}{2|l|},$$

$$\sum_{0<|l|\le L} \left( \sum_{t=1}^{u} |c_t(l)|^2 \right)^{1/2} \le 3u^{1/2} \log(eL),$$

so that (4.7) yields

$$\frac{u\beta}{2} \le 3(2CLu)^{1/2} \log(eL),$$

and indeed

$$|H(r,\beta)| \le 144C \log^2(2e/\beta)\beta^{-3}.$$

Since $H(1,\beta) \subseteq H(2,\beta) \subseteq \dots$, it is clear that

$$\left| \bigcup_{r\ge 1} H(r,\beta) \right| \le 144C \log^2(2e/\beta)\beta^{-3}. \tag{4.8}$$

For $x \in B_\delta(\mathcal{S})$, there is an interval $I$ and an integer $r$ such that

$$\frac{Z(N,I,x)}{N} < m(I) - \delta + \epsilon \quad (N \ge r).$$

Hence

$$B_\delta(\mathcal{S}) \subset \bigcup_{r\ge 1} H(r, \delta - \epsilon). \tag{4.9}$$

Here $\epsilon$ is arbitrary, $0 < \epsilon < \delta$. Theorem 6(i) follows on combining (4.8) and (4.9), and letting $\epsilon$ tend to 0.

Now let $x_1, \dots, x_u$ be distinct points of $H_I(\mathcal{S})$ (if it is a nonempty set). We return to our basic inequality (4.4), in which we now have

$$I_t = I, \quad c_t(l) = c(l), \quad \beta = m(I),$$

and recalling (4.3),

$$|c(l)| \le \frac{3\beta}{2} \qquad (l \ne 0).$$

Thus

$$\left| \sum_{0<|l|\le L} \sum_{k=1}^{N} \sum_{t=1}^{u} c(l) \exp(l a_k x_t) \right| \ge \frac{Nu\beta}{2}.$$

Write

$$d_s = \sum_{\substack{a_k l = s \\ 0<|l|\le L,\ k\le N}} c(l),$$

$$f_s = \sum_{\substack{a_k l = s \\ 0<|l|\le L,\ k\le N}} 1.$$

Clearly $|d_s| \le \frac{3}{2}\beta f_s$. Suppose that $N$ satisfies $n_N \le CN$. We have

$$\left| \sum_{|s|\le CLN} \sum_{t=1}^{u} d_s \exp(s x_t) \right| \ge \frac{Nu\beta}{2}.$$

Cauchy's inequality gives

$$\sum_{|s|\le CLN} |d_s|^2 \sum_{|s|\le CLN} \left| \sum_{t=1}^{u} \exp(s x_t) \right|^2 \ge \frac{1}{4} N^2 u^2 \beta^2. \tag{4.10}$$

Now

$$\sum_{|s|\le CLN} |d_s|^2 \le \frac{9}{4}\beta^2 \sum_{|s|\le CLN} f_s^2.$$

This last sum is simply $2M$, where $M$ is the number of solutions to

$$l a_k = m a_r \qquad 1 \le l,\ m \le L,\ 1 \le k, r \le N.$$

A trivial bound for $M$ is $NL^2$. We can obtain a different bound by noting that for fixed $l$, $m$ we must have $a_k \equiv 0 (\operatorname{mod} m/(m,l))$ and there are $\le CN(m,l)/m$ solutions to this. This yields

$$\begin{aligned} M &\le CN \sum_{1\le l,m\le L} \frac{(m,l)}{m} \\ &\le CN \sum_{d\le L} d \left(\frac{L}{d}\right) \sum_{m\le L/d} \frac{1}{md} \\ &\le CNL \sum_{d\le L} \frac{1}{d} \sum_{m\le L/d} \frac{1}{m} \\ &\le CNL(\log(eL))^2, \end{aligned}$$

and

$$\sum_{|s|\le CLN} |d_s|^2 \le \frac{9}{2}\beta^2 \min\left(NL^2, CNL(\log(eL))^2\right).$$

As for the other factor on the left-hand side of (4.10), we have

$$\sum_{|s|\le CLN}\left|\sum_{t=1}^{u}\exp(sx_t)\right|^2 \le (2+\epsilon)CLNu$$

for sufficiently large $N$, by Lemma 7. We conclude that

$$18(2+\epsilon)Cu(LN)^2\min\left(L, C(\log(eL))^2\right) \ge (Nu)^2.$$

Since $\epsilon$ is arbitrary, this gives the stated result. □

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