THE ZEROS OF A QUADRATIC FORM AT SQUARE-FREE POINTS

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ABSTRACT. Let $F(x_1, \ldots, x_n)$ be a nonsingular indefinite quadratic form, $n = 3$ or 4. Results are obtained on the number of solutions of

$$F(x_1, \ldots, x_n) = 0$$

with $x_1, \ldots, x_n$ square-free, in a large box of side P. It is convenient to count solutions with weights. Let

$$R(F, w) = \sum_{F(\boldsymbol{x})=0} \mu^2(\boldsymbol{x})\, w\left(\frac{\boldsymbol{x}}{P}\right)$$

where w is infinitely differentiable with compact support and vanishes if any $x_i = 0$, while

$$\mu^2(\boldsymbol{x}) = \mu^2(|x_1|) \ldots \mu^2(|x_n|).$$

It is assumed that F is *robust* in the sense that

$$\det M_1 \ldots \det M_n \neq 0,$$

where M_i is the matrix obtained by deleting row i and column i from the matrix M of F. In the case $n = 3$, there is the further hypothesis that $-\det M_1$, $-\det M_2$, $-\det M_3$ are not squares. It is shown that $R(F, w)$ is asymptotic to

$$e_n \sigma_\infty(F, w) \rho^*(F) P^{n-2} \log P,$$

where $e_n = 1$ for $n = 4$, $e_n = \frac{1}{2}$ for $n = 3$. Here $\sigma_\infty(F, w)$ and $\rho^*(F)$ are respectively the singular integral and the singular series associated to the problem. The method is adapted from the approach of Heath-Brown to the corresponding problem with $x_1, \ldots, x_n$ unrestricted integer variables.

1. INTRODUCTION

Let $F(\boldsymbol{x}) = F(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ $(a_{ij} = a_{ji} \in \mathbb{Z})$ be a nonsingular indefinite quadratic form, $n \geq 3$. Let $M = [a_{ij}]$, $D = \det(M)$. We are concerned here with the asymptotics of the square-free solutions $\boldsymbol{x} \in \mathbb{Z}^n$, of

$$F(\boldsymbol{x}) = 0. \tag{1.1}$$

As in [1], let

$$\pi_{\boldsymbol{y}} = y_1 \cdots y_n \quad (\boldsymbol{y} \in \mathbb{R}^n).$$

For $\boldsymbol{x} \in \mathbb{Z}^n$, let

$$\mu(\boldsymbol{x}) = \begin{cases} 0 \text{ if } \pi_{\boldsymbol{x}} = 0 \\ \mu(|x_1|) \ldots \mu(|x_n|) & \text{if } \pi_{\boldsymbol{x}} \neq 0. \end{cases}$$

A square-free solution of (1.1) is a solution having $\mu(\boldsymbol{x}) \neq 0$.

Solutions of (1.1) will be weighted, as in [1], by a function $w\left(\frac{\boldsymbol{x}}{P}\right)$, where the positive parameter P tends to infinity. We assume throughout that

(i) w is infinitely differentiable with compact support;
(ii) $w(\boldsymbol{x}) = 0$ whenever $\pi_{\boldsymbol{x}} = 0$,

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(iii) $w(\boldsymbol{x}) \geq 0$, and $w(\boldsymbol{x}) > 0$ for some real solution $\boldsymbol{x}$ of (1.1).

Our object of study is

$$R(F, w) = \sum_{F(\boldsymbol{x})=0} \mu^2(\boldsymbol{x}) w\left(\frac{\boldsymbol{x}}{P}\right).$$

An asymptotic formula for $R(F, w)$ was obtained in [1] in the cases

(a) $n \geq 5$,
(b) $n = 4$; D not a square.

The method used was an elaboration of that of Heath-Brown [4], whose objective was to obtain an asymptotic formula for

$$N(F, w) = \sum_{F(\boldsymbol{w})=0} w\left(\frac{\boldsymbol{x}}{P}\right).$$

Besides the cases (a), (b), Heath-Brown also successfully treated $N(F, w)$ in the more difficult cases

(c) $n = 4$; D a square,
(d) $n = 3$.

In the present paper, I treat $R(F, w)$ for the cases (c), (d). Some restrictions are imposed on F.

Let M_j be the matrix obtained by deleting row j and column j of M. We say that F is **robust** if

$$\det(M_1)\ldots\det(M_n) \neq 0. \tag{1.2}$$

Our results will apply to robust forms, with a further restriction when $n = 3$.

In order to state the asymptotic formulae, we define the singular integral by

$$\sigma_\infty(F, w) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon} \int_{|F(\boldsymbol{x})|\leq\epsilon} w(\boldsymbol{x})d\boldsymbol{x},$$

where $\int \ldots d\boldsymbol{x}$ denotes integration over $\mathbb{R}^n$ with respect to Lebesgue measure. Under the conditions (i)–(iii), $\sigma_\infty(F, w)$ is positive ([4], Theorem 3).

The singular series for our problem is

$$\rho^*(F) = \prod_p \left(1 - \frac{1}{p}\right)\rho_p.$$

Here ρ_p is given by

$$\rho_p = \lim_{\nu\to\infty} p^{-\nu(n-1)} \#\{\boldsymbol{x} \ (\mathrm{mod}\ p^\nu) : F(\boldsymbol{x}) \equiv 0 \ (\mathrm{mod}\ p^\nu),\ p^2 \nmid x_1, \ldots, p^2 \nmid x_n\}.$$

Thus ρ_p is the p-adic density of solutions of $F = 0$ 'square-free with respect to p'.

Theorem 1. *Let $n = 4$, let D be a square and suppose that F is robust. Then*

$$R(F, w) = \sigma_\infty(F, w)\rho^*(F)P^2 \log P + O(P^2 \log P(\log\log P)^{-1+\epsilon}).$$

As usual, ϵ is an arbitrary positive number, supposed sufficiently small. Constants implied by 'O' and '$\ll$' may depend on F, w and ϵ. Any other dependence will be shown explicitly.

Theorem 2. *Let $n = 3$ and suppose that F is robust. Suppose further that none of $-\det M_1$, $-\det M_2$, $-\det M_3$ is a square. Then*

$$R(F, w) = \frac{1}{2}\,\sigma_\infty(F, w)\rho^*(F)P\log P + O(P\log P(\log\log P)^{-1/2}).$$

The following propositions give information about $\rho^*(F)$.

Proposition 1. *Let F be nonsingular (if $n = 4$) and robust (if $n = 3$).*

(a) if $\rho_p > 0$ for every prime p, then $\rho^*(F) > 0$.
(b) if the congruence

$$F(\boldsymbol{x}) \equiv 0 \pmod{(2D)^5}$$

has a solution with $p^2 \nmid x_1, \dots, p^2 \nmid x_n$ whenever $p \mid 2D$, then $\rho^*(F) > 0$.

Proposition 2. *If $n = 3$ and F is not robust, then $\rho^*(F) = 0$.*

As an example for Proposition 2, it is a simple exercise to show that

$$P \ll \#\{\boldsymbol{x} : \mu(\boldsymbol{x}) \neq 0,\ P \leq x_j < 2P,\ F_0(\boldsymbol{x}) = 0\} \ll P$$

for the ternary form $F_0(\boldsymbol{x}) = 2x_1x_2 - 2x_3^2$. The conclusion of Theorem 2 clearly extends to F_0! In fact, I conjecture that for a non-robust ternary quadratic form F and a given w, there is an asymptotic formula

$$R(F, w) \sim c(F, w)P$$

with $c(F, w) > 0$, precisely when $w > 0$ at some point of a certain set $E = E(F)$ of zeros of F. In the example,

$$E = \{(t, t, \pm t) : t \neq 0\}.$$

Before outlining the proofs of Theorems 1 and 2, we recall some notations from [1] and [4]. We write, for $\boldsymbol{c} \in \mathbb{Z}^n$,

$$S_{q,F}(\boldsymbol{c}) = S_q(\boldsymbol{c}) = \sum_{a=1}^{q}{}^{*} \sum_{\boldsymbol{b} \pmod q} e_q(aF(\boldsymbol{b}) + \boldsymbol{c}\cdot\boldsymbol{b}).$$

As usual, the asterisk indicates $(a, q) = 1$, while

$$\boldsymbol{c}\cdot\boldsymbol{b} = c_1b_1 + \cdots + c_nb_n, \quad e(\theta) = e^{2\pi i\theta},\ e_q(m) = e\left(\frac{m}{q}\right).$$

The symbols $\boldsymbol{d}$ and $\boldsymbol{t}$ are reserved for points in $\mathbb{Z}^n$ with positive square-free coordinates. Let

$$F_{\boldsymbol{d}}(\boldsymbol{x}) = F(d_1^2x_1, \dots, d_n^2x_n)$$

and similarly for $w_{\boldsymbol{d}}(\boldsymbol{x})$. We write

$$S_q(\boldsymbol{d}, \boldsymbol{c}) = S_{q,F_{\boldsymbol{d}}}(\boldsymbol{c}).$$

It is convenient to write $\boldsymbol{d} \mid m$ as an abbreviation for

$$d_1 \mid m, \dots, d_n \mid m.$$

Further, let

$$|\boldsymbol{y}| = \max(|y_1|, \dots, |y_n|).$$

Let $h(x,y)$ $(x>0, y\in\mathbb{R})$ be the smooth function that occurs in Theorem 1 and 2 of [4]. We recall that $h(x,y)$ is nonzero only for $x\le \max(1,2|y|)$. It is shown in [4, Theorem 2] that

$$N(F,w) = c_P P^{-2}\sum_{\boldsymbol{c}\in\mathbb{Z}^n}\sum_{q=1}^{\infty} q^{-n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}),$$

where

$$c_P = 1 + O_N(P^{-N}) \text{ for every } N>0, \tag{1.3}$$

and

$$I_{q,F,w}(\boldsymbol{c}) = I_q(\boldsymbol{c}) = \int_{\mathbb{R}^n} w\left(\frac{\boldsymbol{x}}{P}\right) h\left(\frac{q}{P}, \frac{F(\boldsymbol{x})}{P^2}\right) e_q(-\boldsymbol{c}\cdot\boldsymbol{x}) d\boldsymbol{x}.$$

Clearly $I_{q,F,w}(\boldsymbol{c})$ is nonzero only for $q \ll P$.

As noted in [1],

$$I_{q,F_{\boldsymbol{d}},w_{\boldsymbol{d}}}(\boldsymbol{c}) = \frac{1}{\pi_{\boldsymbol{d}}^2} I_q\left(\frac{c_1}{d^2}, \ldots, \frac{c_n}{d^2}\right). \tag{1.4}$$

Thus

$$N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}}) = \frac{c_P}{\pi_{\boldsymbol{d}}^2 P^2} \sum_{\boldsymbol{c}\in\mathbb{Z}^n}\sum_{q=1}^{\infty} \frac{S_q(\boldsymbol{d},\boldsymbol{c})}{q^n} I_q\left(\frac{c_1}{d_1^2}, \ldots, \frac{c_n}{d_n^2}\right). \tag{1.5}$$

Let

$$z = z(P) = \frac{1}{7}\log\log P,$$
$$Q(z) = \prod_{p<z} p.$$

For $\boldsymbol{x}\in\mathbb{Z}^n$, $\pi_{\boldsymbol{x}}\ne 0$, let

$$f_z(\boldsymbol{x}) = \begin{cases} 1 & \text{if } p^2 \nmid x_j \text{ for } p<z \text{ and } j=1,\ldots,n \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$f_z(\boldsymbol{x}) \ge \mu^2(\boldsymbol{x}) \ge f_z(\boldsymbol{x}) - \sum_{\substack{p\ge z\\ p^2\mid x_1}} 1 - \cdots - \sum_{\substack{p\ge z\\ p^2\mid x_n}} 1.$$

Multiplying by $w\left(\frac{\boldsymbol{x}}{P}\right)$ and summing over $\boldsymbol{x}\in\mathbb{Z}^n$ with $F(\boldsymbol{x})=0$,

$$\sum_{F(\boldsymbol{x})=0} f_z(\boldsymbol{x}) w\left(\frac{\boldsymbol{x}}{P}\right) \ge R(F,w) \ge \sum_{F(\boldsymbol{x})=0} f_z(\boldsymbol{x}) w\left(\frac{\boldsymbol{x}}{P}\right) - (S_1(z)+\cdots+S_n(z)). \tag{1.6}$$

Here

$$S_j(X) = \sum_{\substack{p\ge X,\ p^2\mid x_j\\ F(\boldsymbol{x})=0}} w\left(\frac{\boldsymbol{x}}{P}\right). \tag{1.7}$$

We note that

$$f_z(\boldsymbol{x}) = \sum_{\substack{d_1^2\mid x_1\\ d_1\mid Q(z)}} \cdots \sum_{\substack{d_n^2\mid x_n\\ d_n\mid Q(z)}} \mu(\boldsymbol{d}),$$

so that

$$\sum_{F(\boldsymbol{x})=0} f_z(\boldsymbol{x}) w\left(\frac{\boldsymbol{x}}{P}\right) = \sum_{F(\boldsymbol{x})=0} w\left(\frac{\boldsymbol{x}}{P}\right) \sum_{\substack{d_1^2 \mid x_1, \ldots, d_n^2 \mid x_n \\ \boldsymbol{d} \mid Q(z)}} \mu(\boldsymbol{d}) \tag{1.8}$$

$$= \sum_{\boldsymbol{d} \mid Q(z)} \mu(\boldsymbol{d}) N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}}).$$

We can express $S_j(X)$ somewhat similarly. Take for example $j = 1$ and write $\boldsymbol{d}(p) = (p, 1, \ldots, 1)$,

$$F_p = F_{\boldsymbol{d}(p)}\ ,\ w_p = w_{\boldsymbol{d}(p)}.$$

Then

$$S_1(X) = \sum_{p \geq X} N(F_p, w_p). \tag{1.9}$$

Our plan is to adapt [4] so as to evaluate $N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}})$ via (1.5), making the error explicit in $\boldsymbol{d}$, and then apply this to the last expression in (1.8) and to $N(F_p, w_p)$. The contribution to $S_1(z)$ from $p \geq P^\epsilon$ will receive a more elementary treatment, similar to [1, Proposition 1].

In conclusion, I point out a refinement of a theorem in [1] due to Blomer [2]. Let $R(m)$ be the number of representations of m as a sum of 3 squarefree integers. If the square-free kernel of m is at least m^δ, for a positive constant δ, and $m \equiv 1, 3$ or $6 \pmod 8$, then Blomer obtains

$$R(m) = c_\infty \mathfrak{S}(m) m^{1/2} + O(m^{(1-\gamma)/2}),\ \gamma = \gamma(\delta) > 0. \tag{1.10}$$

Here c_∞ is the singular integral and $\mathfrak{S}(m)$ the singular series,

$$m^{-\epsilon} \ll \mathfrak{S}(m) \ll m^\epsilon.$$

In [1], (1.10) is obtained only for square-free m.

2. Some exponential integrals, exponential sums and Dirichlet series

From now on we assume that $n = 3$ or 4, and the determinant of F is a square for $n = 4$.

It suffices to prove Theorems 1 and 2 for weight functions w with the following property: there exists a positive number $\ell = \ell(F, w)$ such that, whenever $(x_0, y) \in \operatorname{supp}(w)$, we have

$$\frac{\partial F}{\partial x}(x, y) \gg 1 \quad (|x - x_0| \leq \ell)$$

and F has exactly one zero (x, y) with $|x - x_0| \leq \ell$. We shall assume that w has this property. The deduction of the general case of Theorems 1 and 2 is carried out by a simple procedure given on page 179 of [4].

As noted on page 180 of [4],

$$I_q(\boldsymbol{v}) = P^n I_r^*(\boldsymbol{v}) \quad (r = P^{-1} q), \tag{2.1}$$

where

$$I_r^*(\boldsymbol{v}) = \int_{\mathbb{R}^n} w(\boldsymbol{x}) h(r, F(\boldsymbol{x})) e_r(-\boldsymbol{v} \cdot \boldsymbol{x}) d\boldsymbol{x}. \tag{2.2}$$

For $\boldsymbol{v} = \boldsymbol{0}$, we have

$$I_r^*(\boldsymbol{0}) = \sigma_\infty(F, w) + O_N(r^N) \tag{2.3}$$

for any $N > 0$, provided that $r \ll 1$ [4, Lemma 13]. Consequently

$$I_q(\boldsymbol{0}) = P^n\{\sigma_\infty(F, w) + O_N((q/P)^N)\} \tag{2.4}$$

for $q \ll P$.

By combining the conclusions of [4, Lemmas 14, 15, 16, 18, 19, 22], we arrive at the following bounds:

$$I_r^*(\boldsymbol{v}) \ll 1, \tag{2.5}$$

$$r\,\frac{\partial I_r^*(\boldsymbol{v})}{\partial r} \ll 1, \tag{2.6}$$

$$I_q(\boldsymbol{v}) \ll P^n, \tag{2.7}$$

$$q\,\frac{\partial I_q(\boldsymbol{c})}{\partial q} \ll P^n, \tag{2.8}$$

$$I_r^*(\boldsymbol{v}) \ll_N r^{-1}\,|\boldsymbol{v}|^{-N} \quad (N \geq 1) \tag{2.9}$$

$$I_q(\boldsymbol{v}) \ll_N P^{n+1} q^{-1} |\boldsymbol{v}|^{-N} \quad (N \geq 1), \tag{2.10}$$

$$I_r^*(\boldsymbol{v}) \ll (r^{-2}|\boldsymbol{v}|)^{\epsilon}(r^{-1}|\boldsymbol{v}|)^{1-n/2}, \tag{2.11}$$

$$I_q(\boldsymbol{v}) \ll P^n \left(\frac{P^2|\boldsymbol{v}|}{q^2}\right)^{\epsilon} \left(\frac{P|\boldsymbol{v}|}{q}\right)^{1-n/2}, \tag{2.12}$$

$$q\,\frac{\partial}{\partial q}\,I_q(\boldsymbol{v}) \ll P^n \left(\frac{P^2|\boldsymbol{v}|}{q^2}\right)^{\epsilon} \left(\frac{P|\boldsymbol{v}|}{q}\right)^{1-n/2}. \tag{2.13}$$

Lemma 1. *For any $K > 1$,*

$$\int_0^\infty r^{-1} I_r^*(\boldsymbol{v}) dr \ll_K |\boldsymbol{v}|^{-K} \quad (|\boldsymbol{v}| > 1), \tag{2.14}$$

$$\int_0^\infty r^{-1} I_r^*(\boldsymbol{v}) dr \ll \log\left(\frac{2}{|\boldsymbol{v}|}\right) \quad (|\boldsymbol{v}| \leq 1), \tag{2.15}$$

$$\int_0^\infty q^{-1} I_q(\boldsymbol{v}) dq \ll_M P^n |\boldsymbol{v}|^{-K} \quad (|\boldsymbol{v}| > 1), \tag{2.16}$$

$$\int_0^\infty q^{-1} I_q(\boldsymbol{v}) dq \ll P^n \log\left(\frac{2}{|\boldsymbol{v}|}\right) \quad (|\boldsymbol{v}| \leq 1). \tag{2.17}$$

Proof. In view of (2.1), it suffices to prove (2.14) and (2.15). Suppose first that $|\boldsymbol{v}| > 1$. We use (2.11) for the range

$$r \leq |\boldsymbol{v}|^{-N/2}$$

and (2.9) for the remaining range. Thus

$$\begin{aligned}
\int_0^\infty r^{-1} I_r^*(\boldsymbol{v}) dr &\ll |\boldsymbol{v}|^{\epsilon+1-n/2} \int_0^{|\boldsymbol{v}|^{-N/2}} r^{n/2-1-2\epsilon} dr \\
&\quad + |\boldsymbol{v}|^{-N} \int_{|\boldsymbol{v}|^{-N/2}} r^{-2} dr \\
&\ll |\boldsymbol{v}|^{\epsilon+1-n/2-N/2\,(n/2-2\epsilon)} + |\boldsymbol{v}|^{-N/2} \\
&\ll_K |\boldsymbol{v}|^{-K}
\end{aligned}$$

for a suitable choice of $N = N(K, \epsilon)$.

Now suppose that $|\boldsymbol{v}| \le 1$. We use (2.11) for the range $r \le |\boldsymbol{v}|$, (2.5) for the range

$$|\boldsymbol{v}| < r \le |\boldsymbol{v}|^{-1},$$

and (2.9) with $N = 1$ for the remaining range. Thus

$$\begin{aligned}\int_0^\infty r^{-1} I_r^*(\boldsymbol{v}) dr &\ll |\boldsymbol{v}|^{\epsilon+1-n/2} \int_0^{|\boldsymbol{v}|} r^{n/2-1-2\epsilon} dr \\ &\quad + \int_{|\boldsymbol{v}|}^{|\boldsymbol{v}|^{-1}} r^{-1} dr + |\boldsymbol{v}|^{-1} \int_{|\boldsymbol{v}|^{-1}}^\infty r^{-2} dr \\ &\ll |\boldsymbol{v}|^{1-\epsilon} + 2 \log\left(\frac{1}{|\boldsymbol{v}|}\right) + 1 \ll \log\left(\frac{2}{|\boldsymbol{v}|}\right).\end{aligned}$$

We now turn to estimates for $S_q(\boldsymbol{d}, \boldsymbol{c})$. Let $M_{\boldsymbol{d}}$ be the matrix

$$M_{\boldsymbol{d}} = [d_i^2 d_j^2 a_{ij}].$$

Thus

$$\det M_{\boldsymbol{d}} = \pi_{\boldsymbol{d}}^4 \det(M), \quad \det M_{\boldsymbol{d}}^{-1} = \frac{(\det(M))^{-1}}{\pi_{\boldsymbol{d}}^4}. \tag{2.18}$$

Writing $M^{-1} = \frac{1}{\det(M)}[b_{ij}]$, so that $b_{ij} \in \mathbb{Z}$, we note that

$$M_{\boldsymbol{d}}^{-1} = \frac{1}{\det(M)} \left[\frac{b_{ij}}{d_i^2 d_j^2}\right]. \tag{2.19}$$

We write $M_{\boldsymbol{d}}^{-1}(\boldsymbol{x})$ for the quadratic form, with rational coefficients, whose matrix is $M_{\boldsymbol{d}}^{-1}$. Let $\Delta = 2|\det M|$. When $p \nmid \pi_{\boldsymbol{d}}\Delta$, we may think of $M_{\boldsymbol{d}}^{-1}(\boldsymbol{x})$ as being defined modulo p.

We recall that, for any nonsingular form F,

$$S_q(\boldsymbol{d}, \boldsymbol{c}) \ll q^{1+n/2}(d_1^2, q) \dots (d_n^2, q) \tag{2.20}$$

[1, Lemma 9]. We need a slight generalization of (2.20). Let $\boldsymbol{c}(a)$ be a vector in $\mathbb{Z}^n$ for every $a = 1, \dots, q, (a, q) = 1$. Let

$$S_{\boldsymbol{d}} = \sum_{a=1}^{q}{}^{*} \sum_{\boldsymbol{b} \,(\mathrm{mod}\, q)} e_q(aF_{\boldsymbol{d}}(\boldsymbol{b}) + \boldsymbol{c}(a) \cdot \boldsymbol{b}).$$

Then

$$S_{\boldsymbol{d}} \ll q^{1+n/2}(d_1^2, q) \dots (d_n^2, q).$$

To see this, Cauchy's inequality yields

$$|S_{\boldsymbol{d}}|^2 \le \phi(q) \sum_{a=1}^{q}{}^{*} \sum_{\boldsymbol{u},\boldsymbol{v} \,(\mathrm{mod}\, q)} e_q(a(F_{\boldsymbol{d}}(\boldsymbol{u}) - F_{\boldsymbol{d}}(\boldsymbol{v})) + \boldsymbol{c}(\boldsymbol{a}) \cdot (\boldsymbol{u} - \boldsymbol{v})).$$

Substitute $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, so that

$$\begin{aligned}&e_q(a(F_{\boldsymbol{d}}(\boldsymbol{u}) - F_{\boldsymbol{d}}(\boldsymbol{v})) + \boldsymbol{c}(\boldsymbol{a}) \cdot (\boldsymbol{u} - \boldsymbol{v})) \\ &= e_q(aF(\boldsymbol{w}) + \boldsymbol{c}(a) \cdot \boldsymbol{w})\, e_q(a\boldsymbol{v} \cdot \nabla F(\boldsymbol{w})).\end{aligned}$$

The summation over $\boldsymbol{v}$ will now produce a contribution of zero unless q divides $\nabla F_{\boldsymbol{d}}(\boldsymbol{w}) = 2M_{\boldsymbol{d}}\boldsymbol{w}$. We have

$$|S_{\boldsymbol{d}}|^2 \leq q^n \phi(q)^2 \sum_{\substack{\boldsymbol{w} \pmod q \\ 2M_{\boldsymbol{d}}\boldsymbol{w} \equiv 0 \pmod q}} 1.$$

We may now complete the proof with the argument used for [1, Lemma 9].

Since

$$S_{uv}(\boldsymbol{d}, \boldsymbol{c}) = S_u(d, \bar{v}\boldsymbol{c}) S_v(d, \bar{u}\boldsymbol{c}) \tag{2.21}$$

where $u\bar{u} \equiv 1 \pmod v$, $v\bar{v} \equiv 1 \pmod u$ [4, Lemma 23], we can do most of our work for prime powers q.

For $n = 4$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) \neq 0$, we have

$$\sum_{q \leq X} |S_q(\boldsymbol{d}, \boldsymbol{c})| \ll \pi_{\boldsymbol{d}}^2 X^{7/2+\epsilon}(|\boldsymbol{c}| + 1)^{\epsilon} \tag{2.22}$$

[1, Lemma 10]. To get results that play a comparable role when $n = 4$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) = 0$ or $n = 3$, we use the Dirichlet series

$$\zeta(s, \boldsymbol{d}, \boldsymbol{c}) = \sum_{q=1}^{\infty} q^{-s} S_q(\boldsymbol{c}) \quad (\sigma > 2 + n/2)$$

$$= \prod_p \left\{ \sum_{u=0}^{\infty} p^{-us} S_{p^u}(\boldsymbol{d}, \boldsymbol{c}) \right\}$$

[4, p. 194]. Bounds for those Euler factors for which $p \mid \pi_{\boldsymbol{d}}$ will require extra work compared to the analysis on pages 194–5 of [4]. If we write

$$\tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) = \prod_{p \mid \pi_{\boldsymbol{d}}} \sum_{u=0}^{\infty} p^{-u\sigma} |S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|, \tag{2.23}$$

we see that the analysis in question gives

(i) for $n = 3$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) \neq 0$,

$$\zeta(s, \boldsymbol{d}, \boldsymbol{c}) = L(s - 2, \chi_{\boldsymbol{d},\boldsymbol{c}}) \nu(s, \boldsymbol{d}, \boldsymbol{c}) \tag{2.24}$$

where

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) = \prod (1 - \chi_{\boldsymbol{d},\boldsymbol{c}}(p) p^{2-s}) \left\{ \sum_{u=0}^{\infty} p^{-us} S_{p^u}(\boldsymbol{c}) \right\},$$

and $\chi_{\boldsymbol{d},\boldsymbol{c}}$ is a character satisfying

$$\chi_{\boldsymbol{d},\boldsymbol{c}}(p) = \left(\frac{-\det(M_{\boldsymbol{d}}) M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})}{p} \right). \tag{2.25}$$

We note that $\chi_{\boldsymbol{d},\boldsymbol{c}}$ (if not trivial) is a character to modulus $4\Delta\pi_{\boldsymbol{d}}^4 \mid M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})|$. Moreover,

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) \ll |\boldsymbol{c}|^{\epsilon} \tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \quad \left(\sigma \geq \frac{17}{6} + \epsilon \right). \tag{2.26}$$

(ii) for $n = 3$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) = 0$,

$$\zeta(s, \boldsymbol{d}, \boldsymbol{c}) = \zeta(2s - 5) \nu(s, \boldsymbol{d}, \boldsymbol{c}), \tag{2.27}$$

with

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) = \prod_p (1 - p^{5-2s}) \left\{ \sum_{u=0}^{\infty} p^{-us} S_{p^u}(\boldsymbol{d}, \boldsymbol{c}) \right\}.$$

Moreover,

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) \ll \tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \quad \left(\sigma \geq \frac{17}{6} + \epsilon\right). \tag{2.28}$$

(iii) for $n = 4$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) = 0$, we find that

$$\zeta(s, \boldsymbol{d}, \boldsymbol{c}) = L(s-3, \chi_{\boldsymbol{d}}) \nu(s, \boldsymbol{d}, \boldsymbol{c}),$$

where

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) = \prod_p (1 - \chi_{\boldsymbol{d}}(p) p^{3-s}) \left\{ \sum_{u=0}^{\infty} p^{-us} S_{p^u}(\boldsymbol{d}, \boldsymbol{c}) \right\},$$

with a character $\chi_{\boldsymbol{d}}$ satisfying

$$\chi_{\boldsymbol{d}}(p) = \left(\frac{\det M_{\boldsymbol{d}}}{p} \right).$$

Since $\det M_{\boldsymbol{d}}$ is a square, we take the trivial character, and write

$$\zeta(s, \boldsymbol{d}, \boldsymbol{c}) = \zeta(s-3) \nu(s, \boldsymbol{d}, \boldsymbol{c}). \tag{2.29}$$

Moreover,

$$\nu(s, \boldsymbol{d}, \boldsymbol{c}) \ll \tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \quad \left(\sigma \geq \frac{7}{2} + \epsilon\right). \tag{2.30}$$

For any $\boldsymbol{d}$, we write $t_j = t_j(\boldsymbol{d})$ for the product of those primes dividing exactly j of $d_1, \ldots, d_n$. Evidently,

$$\pi_{\boldsymbol{d}} = t_1 t_2^2 \ldots t_n^n.$$

We also write

$$A(\boldsymbol{d}) = \begin{cases} \pi_{\boldsymbol{d}}^{5\epsilon} \, t_2^2 \, (t_3 t_4)^4 & (n = 4) \\ \pi_{\boldsymbol{d}}^{5\epsilon} \, t_2^{5/2} \, t_3^4 & (n = 3). \end{cases} \tag{2.31}$$

Let us write $\alpha_3 = 17/6$, $\alpha_4 = 7/2$.

Lemma 2. *(i) Let F be nonsingular. Then*

$$\tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \ll \pi_{\boldsymbol{d}}^{2+\epsilon} \quad (\sigma \geq \alpha_n + \epsilon).$$

(ii) Let $\sigma \geq n - \epsilon$. Suppose that F is nonsingular (if $n = 4$) and robust (if $n = 3$). Then

$$\tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \ll A(\boldsymbol{d}).$$

Proof. (i) For $n = 3$, (2.20) yields

$$\frac{S_{p^u}(\boldsymbol{d}, \boldsymbol{c})}{p^{\sigma u}} \ll p^{-(1/3+\epsilon)u} (d_1^2, p^2)(d_2^2, p^2)(d_3^2, p^2),$$

$$1 + \sum_{u \geq 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} \ll (d_1^2, p^2)(d_2^2, p^2)(d_3^2, p^2),$$

$$\tau_{\boldsymbol{d}}(\boldsymbol{c}, \sigma) \ll \pi_{\boldsymbol{d}}^{\epsilon} \prod_{p \mid \pi_{\boldsymbol{d}}} (d_1^2, p^2)(d_2^2, p^2)(d_3^2, p^2)$$

$$= \pi_{\boldsymbol{d}}^{2+\epsilon}.$$

The argument is similar for $n = 4$.

(ii) For $n = 4$, (2.20) yields

$$1 + \sum_{u\ge 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} \ll 1 + p^{-(1-\epsilon)}(d_1^2, p)\dots(d_4^2, p) + p^{-(2-2\epsilon)}(d_1^2, p^2)\cdots(d_4^2, p^2). \tag{2.32}$$

If p divides t_1, then

$$1 + \sum_{u\ge 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} \ll p^{2\epsilon}.$$

If p divides t_2, then

$$1 + \sum_{u\ge 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} \ll p^{2+2\epsilon}.$$

If p divides t_3t_4, then

$$1 + \sum_{u\ge 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} \ll p^{4+4\epsilon} + \sum_{u\ge 5} p^{8-(1-\epsilon)u} \ll p^{4+4\epsilon}.$$

Here we use the trivial bound ($u \le 4$) and (2.20) ($u \ge 5$). Lemma 2(ii) follows for $n = 4$.

Now let $n = 3$. Suppose that $p \,|\, t_1$; let us say $p \,|\, d_1$. Then for $u \le 4$, and a fixed value of x_1, let us write

$$G(x_2, x_3) = \sum_{j,k=2}^{3} a_{jk} d_j^2 d_k^2 x_j x_k,$$

$$\boldsymbol{h} = \boldsymbol{h}(a) = (aa_{12}d_1^2 d_2^2 x_1 + c_2,\ aa_{13}d_1^2 d_3^2 x_1 + c_3).$$

We have

$$\begin{aligned} aF_{\boldsymbol{d}}(\boldsymbol{x}) \cdot \boldsymbol{c} &\equiv aG(x_2, x_3) + \sum_{k=2}^{3} aa_{1k} d_1^2 d_k^2 x_1 x_k + \boldsymbol{x} \cdot \boldsymbol{c} \\ &\equiv x_1 c_1 + aG(x_2, x_3) + (x_2, x_3) \cdot \boldsymbol{h} \pmod{p^u}, \end{aligned}$$

$$|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|^2 \le p^u \sum_{x_1=1}^{p^u} \left| \sideset{}{^*}\sum_{a=1}^{p^u} \sum_{\boldsymbol{y} \pmod{p^u}} e(aG(\boldsymbol{y}) + \boldsymbol{y} \cdot \boldsymbol{h}) \right|^2$$

(by Cauchy's inequality)

$$\ll p^{2u}(p^{2u})^2$$

by the generalization of (2.20) noted above, with n replaced by 2 and F replaced by $F(0, x_2, x_3)$. Hence, applying (2.20) directly for $u \ge 5$,

$$\begin{aligned} 1 + \sum_{u\ge 1} \frac{|S_{p^u}(\boldsymbol{d}, \boldsymbol{c})|}{p^{\sigma u}} &\ll p^{4\epsilon} + \sum_{u\ge 5} p^{2-u(\frac{1}{2}-\epsilon)} \\ &\ll p^{4\epsilon}. \end{aligned}$$

Now let $p \mid t_2$. Then

$$\frac{S_{p^u}(\boldsymbol{d},\boldsymbol{c})}{p^{\sigma u}} \ll \begin{cases} p^{2+2\epsilon} \ (u \le 2) \\ p^{4-u(1/2-\epsilon)} \ (u \ge 3). \end{cases} \tag{2.33}$$

Here we use the trivial bound ($u \le 2$) and (2.20) ($u \ge 3$). Hence

$$1 + \sum_{u \ge 1} \frac{|S_{p^u}(\boldsymbol{d},\boldsymbol{c})|}{p^{\sigma u}} \ll p^{5/2+3\epsilon}.$$

Similarly, if $p \mid t_3$,

$$\frac{S_{p^u}(\boldsymbol{d},\boldsymbol{c})}{p^{\sigma u}} \ll \begin{cases} p^{4+4\epsilon} \ (u \le 4) \\ p^{6-u(\frac{1}{2}-\epsilon)} \ (u \ge 5), \end{cases}$$

$$1 + \sum_{u \ge 1} \frac{|S_{p^u}(\boldsymbol{d},\boldsymbol{c})|}{p^{\sigma u}} \ll p^{4+4\epsilon}.$$

We now complete the proof as above.

The next lemma is useful for singular series calculations.

Lemma 3. *Let F be nonsingular,*

$$\Lambda_p(F) = \sum_{\substack{\boldsymbol{d} \mid p \\ \pi_{\boldsymbol{d}} > 1}} \sum_{u \ge 1} p^{-nu} |S_{p^u}(\boldsymbol{d},\boldsymbol{0})|.$$

Then

$$\Lambda_p(F) \ll \begin{cases} p^{-2} & (n = 4, F \text{ nonsingular}) \\ p^{-3/2} & (n = 3, F \text{ robust}). \end{cases}$$

If $n = 3$ and F is not robust, then $\Lambda_p(F) \ll p^{-1}$.

Proof. Suppose first that $n = 4$. The proof of Lemma 2 (ii) shows that, for $\boldsymbol{d} \mid p$,

$$\sum_{u \ge 1} p^{-nu} |S_{p^u}(\boldsymbol{d},\boldsymbol{0})| \ll \begin{cases} 1 & (\pi_{\boldsymbol{d}} = p) \\ p^2 & (\pi_{\boldsymbol{d}} = p^2) \\ p^4 & (\pi_{\boldsymbol{d}} \ge p^3). \end{cases}$$

Hence

$$\pi_{\boldsymbol{d}}^{-2} \sum_{u \ge 1} p^{-nu} |S_{p^u}(\boldsymbol{d},\boldsymbol{0})| \ll p^{-2},$$

and we obtain the desired bound since $\boldsymbol{d}$ has $O(1)$ values.

The argument for $n = 3$ is similar in the case when F is robust. However, if F is not robust, we have the weaker bound

$$\sum_{u \ge 1} p^{-nu} |S_{p^u}(\boldsymbol{d},\boldsymbol{0})| \ll p \quad (\pi_{\boldsymbol{d}} = p). \tag{2.34}$$

For the left-hand side of (2.34) is

$$\ll p^{-1/2}(d_1^2, p)(d_2^2, p)(d_3^2, p) + p^{-1}(d_1^2, p^2)(d_2^2, p^2)(d_3^2, p^2)$$

from (2.20).

3. Sums of $S_q(\boldsymbol{d},\boldsymbol{c})$ and $S_q(\boldsymbol{d},\boldsymbol{0})q^{-n}$.

Let $e_n = 1$ if $n = 4$ and $e_n = 1/2$ if $n = 3$.

We assume throughout Sections 3 and 4 that F is robust ($n = 3$) and nonsingular ($n = 4$). Define

$$\eta(\boldsymbol{d},\boldsymbol{c}) = \begin{cases} e_n & \text{if } M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) = 0 \\ 1 & \text{if } n = 3 \text{ and } -(\det M_{\boldsymbol{d}})M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) \text{ is a nonzero square} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Define

$$\sigma(\boldsymbol{d},\boldsymbol{c}) = \nu(n,\boldsymbol{d},\boldsymbol{c}).$$

We observe that whenever $\eta(\boldsymbol{d},\boldsymbol{c}) \neq 0$,

$$\sigma(\boldsymbol{d},\boldsymbol{c}) = \prod_p \sigma_p(\boldsymbol{d},\boldsymbol{c}),$$

where

$$\sigma_p(\boldsymbol{d},\boldsymbol{c}) = (1-p^{-1})\sum_{u=0}^{\infty} p^{-nu} S_{p^u}(\boldsymbol{d},\boldsymbol{c}).$$

Lemma 4. *For $X > 1$,*

$$\sum_{q\le X} S_q(\boldsymbol{d},\boldsymbol{c}) = \eta(\boldsymbol{d},\boldsymbol{c})\sigma(\boldsymbol{d},\boldsymbol{c})\,\frac{X^n}{n} + O(X^{\alpha_n+2\epsilon}\pi_{\boldsymbol{d}}^{3+\epsilon}(1+|\boldsymbol{c}|)^{1/2}).$$

Proof. The case $n = 4$, $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) \neq 0$ follows from (2.22), and we exclude this case below.

We recall the version of Perron's formula given in [1, Lemma 13]. Let b, c be positive constants and λ a real constant, $\lambda + c > 1 + b$. For $K > 0$ and complex numbers a_ℓ ($\ell \ge 1$) with $|a_\ell| \le K\ell^b$, write

$$h(s) = \sum_{\ell=1}^{\infty} \frac{a_\ell}{\ell^s} \quad (\sigma > 1 + b);\ \text{then}$$

$$\sum_{\ell\le x} \frac{a_\ell}{\ell^\lambda} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} h(s+\lambda)\,\frac{x^s}{s}\,ds + O\left(\frac{Kx^c}{T}\right) \tag{3.2}$$

whenever $x > 1$, $T > 1$, $x - 1/2 \in \mathbb{Z}$.

For $n = 4$, let $a_\ell = S_\ell(\boldsymbol{d},\boldsymbol{c})$, $b = 3$, $\lambda = 0$, $x = [X] + 1/2$, $T = x^{10}$. According to (2.20), we may take $K \ll \pi_{\boldsymbol{d}}^2$. Recalling (2.29),

$$\begin{aligned}\sum_{q\le X} S_q(\boldsymbol{d},\boldsymbol{c}) &= \frac{1}{2\pi i}\int_{5-iT}^{5+iT} \zeta(s,\boldsymbol{d},\boldsymbol{c})\,\frac{x^2}{s} + O(\pi_{\boldsymbol{d}}^2)\\ &= \frac{1}{2\pi i}\int_{5-iT}^{5+iT} \zeta(s-3)\nu(s,\boldsymbol{d},\boldsymbol{c})\,\frac{x^s}{s}\,ds + O(\pi_{\boldsymbol{d}}^2).\end{aligned}$$

We move the line of integration back to $\sigma = \frac{7}{2} + \epsilon$. On the line segments $[7/2 + \epsilon, 5] \pm iT$,

$$\zeta(s-3) \ll T^{1/4},$$

$$\frac{\nu(s,\boldsymbol{d},\boldsymbol{c})x^s}{s} \ll \pi_{\boldsymbol{d}}^{2+\epsilon}\,T^{-1/2}$$

from (2.30) and Lemma 2 (i). Thus these segments contribute $O(\pi_{\boldsymbol{d}}^{2+\epsilon})$. Since

$$\int_0^U |L(\sigma+it,\chi)|^2 dt \ll k^{1/2} U \quad \left(\frac{1}{2}<\sigma<1\right)$$

for a Dirichlet L-function to modulus k, we have

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+\epsilon+it\right)\nu(s,\boldsymbol{d},\boldsymbol{c})\right| \frac{dt}{1+|t|} \ll \pi_{\boldsymbol{d}}^{2+\epsilon}\log T.$$

Hence the segment $[7/2+\epsilon-iT, 7/2+\epsilon+iT]$ contributes $O(X^{7/2+2\epsilon}\pi_{\boldsymbol{d}}^{2+\epsilon})$. Writing Res for the residue of the integrand at $s=4$, with Res $=0$ if there is no pole,

$$\sum_{q\le X} S_q(\boldsymbol{d},\boldsymbol{c}) = \mathrm{Res} + O(\pi_{\boldsymbol{d}}^{2+\epsilon}X^{7/2+2\epsilon}).$$

Similarly, for $n=3$,

$$\sum_{q\le X} S_q(\boldsymbol{d},\boldsymbol{c}) = \frac{1}{2\pi i}\int_{5-iT}^{5+iT} \zeta(s,\boldsymbol{d},\boldsymbol{c})\,\frac{x^s}{s}\,ds + O(\pi_{\boldsymbol{d}}^2)$$

$$= \frac{1}{2\pi i}\int_{5-iT}^{5+iT} E(s)\nu(s,\boldsymbol{d},\boldsymbol{c})\,\frac{x^s}{s}\,ds + O(\pi_{\boldsymbol{d}}^2),$$

where

$$E(s) = \begin{cases} L(s-2,\chi) & \text{if} \quad M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})\neq 0 \\ \zeta(2s-5) & \text{if} \quad M_{\boldsymbol{d}}^{-1}(\boldsymbol{c}) = 0 \end{cases}$$

and $\chi=\chi_{\boldsymbol{d},\boldsymbol{c}}$ satisfies (2.25). We take χ to be the trivial character if $-\det(M_{\boldsymbol{d}})M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})$ is a nonzero square. Since χ is a character to modulus $k=O(\pi_{\boldsymbol{d}}^4|\boldsymbol{c}|^2)$, a simple hybrid bound [3, Lemma 1] yields

$$E(s) = O((kT)^{1/4})$$
$$= O\left((1+|\boldsymbol{c}|)^{1/2}\pi_{\boldsymbol{d}}T^{1/4}\right)$$

for $\sigma\ge 11/4$, $|t|\le T$.

We move the line of integration back to $\sigma = 17/6+\epsilon$. A slight variant of the preceding argument gives

$$\sum_{q\le X} S_q(\boldsymbol{d},\boldsymbol{c}) = \mathrm{Res} + O\left(X^{17/6+2\epsilon}(1+|\boldsymbol{c}|)^{1/2}\pi_{\boldsymbol{d}}^{3+\epsilon}\right).$$

It now suffices to show that the residue at n is

$$\eta(\boldsymbol{d},\boldsymbol{c})\sigma(\boldsymbol{d},\boldsymbol{c})\,\frac{x^n}{n}.$$

In the case $n=4$, the residue is

$$\nu(4,\boldsymbol{d},\boldsymbol{c})\,\frac{x^4}{4}$$

as required.

For $n=3$, there is no pole unless either $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})=0$ or $M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})\neq 0$ and $\chi_{\boldsymbol{d},\boldsymbol{c}}$ is trivial, that is, $-\det(M_{\boldsymbol{d}})M_{\boldsymbol{d}}^{-1}(\boldsymbol{c})$ is a nonzero square. The residue is $\sigma(\boldsymbol{d},\boldsymbol{c})\,\frac{x^3}{3}$ or $\frac{1}{2}\,\sigma(\boldsymbol{d},\boldsymbol{c})\,\frac{x^3}{3}$ depending on whether the coefficient of $\frac{1}{s-3}$ in the Laurent expansion of the zeta factor is 1 or $\frac{1}{2}$, and the lemma follows.

Lemma 5. *For $X > 1$,*

$$\sum_{q \le X} q^{-n} S_q(\boldsymbol{d}, \boldsymbol{0}) = e_n \sigma(\boldsymbol{d}, \boldsymbol{0}) \log X + O(A(\boldsymbol{d}) + \pi_{\boldsymbol{d}}^{2+\epsilon} X^{\alpha_n - n + 2\epsilon}).$$

Proof. For $n = 4$, we apply (3.2) with a_ℓ, b, x, T, K as in the preceding proof, but now $\lambda = 4$, $c = 1$. This leads to

$$\sum_{q \le X} q^{-4} S_q(\boldsymbol{d}, \boldsymbol{0}) = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \zeta(s+1) \nu(s+4, \boldsymbol{d}, \boldsymbol{0}) \frac{x^s}{s} \, ds + O(\pi_{\boldsymbol{d}}^2).$$

We move the line of integration back to $\sigma = -\frac{1}{2} + \epsilon$. The integrals along segments are $O(\pi_{\boldsymbol{d}}^{2+\epsilon} X^{-1/2+2\epsilon})$ by a variant of the above argument. There is a double pole at 0; the Laurent series of the integrand is

$$\frac{1}{s^2} (1 + as + \cdots)(\nu(4, \boldsymbol{d}, \boldsymbol{0}) + \nu'(4, \boldsymbol{d}, \boldsymbol{0}) s + \cdots)(1 + (\log x) s + \cdots),$$

where a is an absolute constant. The residue is

$$\begin{aligned} &\nu(4, \boldsymbol{d}, \boldsymbol{0})(\log x + a) + \nu'(4, \boldsymbol{d}, \boldsymbol{0}) \\ &= \sigma(\boldsymbol{d}, \boldsymbol{0}) \log X + O\left(\max_{\sigma \ge 4 - \epsilon} \tau_{\boldsymbol{d}}(\boldsymbol{0}, \sigma) + 1 \right). \end{aligned}$$

To get the last estimate, we write $\nu'(4, \boldsymbol{d}, \boldsymbol{0})$ as a contour integral on $|s - 4| = \epsilon$ using Cauchy's formula for a derivative, and apply (2.30). We now complete the proof using Lemma 2 (ii).

For $n = 3$, a similar argument gives

$$\sum_{q \le X} q^{-3} S_q(\boldsymbol{d}, \boldsymbol{0}) = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \zeta(2s+1) \nu(s+3, \boldsymbol{d}, \boldsymbol{0}) \frac{x^s}{s} \, ds + O(\pi_{\boldsymbol{d}}^2).$$

We move the line of integration back to $\sigma = -\frac{1}{6} + \epsilon$, estimating the integrals along line segments as $O(\pi_{\boldsymbol{d}}^{2+\epsilon} X^{-1/6+\epsilon})$. This time the Laurent series at 0 is

$$\frac{1}{2s^2} (1 + 2as + \cdots)(\nu(3, \boldsymbol{d}, \boldsymbol{0}) + \nu'(3, \boldsymbol{d}, \boldsymbol{0}) s + \cdots)(1 + (\log x) s + \cdots)$$

with residue

$$\frac{1}{2} \nu(3, \boldsymbol{d}, \boldsymbol{0})(\log x + 2a) + \frac{1}{2} \nu'(3, \boldsymbol{d}, \boldsymbol{0}),$$

and we complete the proof as before.

4. Evaluation of $N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}})$.

We fix $\boldsymbol{d}$ for the present, with

$$|\boldsymbol{d}| \le P^\epsilon,$$

and write

$$\boldsymbol{c}' = \left(\frac{c_1}{d_1^2}, \ldots, \frac{c_n}{c_n^2} \right).$$

Lemma 6. *We have*

$$\sum_{\substack{\boldsymbol{c} \in \mathbb{Z}^n \\ |\boldsymbol{c}'| > P^\epsilon}} \left| \sum_{q=1}^\infty q^{-n} S_q(\boldsymbol{d}, \boldsymbol{c}) I_q(\boldsymbol{c}') \right| \ll P^n.$$

Proof. We note first that for $A \geq 1$, $R > 1$, $N \geq 2$,

$$\sum_{c > AR} (cA^{-1})^{-N} = A^N \sum_{k=0}^{\infty} \sum_{2^k AR < c \leq 2^{k+1}AR} c^{-N} \tag{4.1}$$

$$\ll A^N \sum_{k=0}^{\infty} 2^{-(N-1)k} A^{-N+1} R^{-N+1} \ll AR^{-N+1}.$$

Taking $A = d_1^2$, $R = P^\epsilon$, we have

$$\sum_{|c_1| > d_1^2 P^\epsilon} (c_1 d_1^{-2})^{-N} \sum_{\max\left(\frac{|c_1|}{d_2^2}, \ldots, \frac{|c_n|}{d_n^2}\right) \leq \frac{c_1}{d_1^2}} 1$$

$$\ll P^{2(n-1)\epsilon} \sum_{|c_1| > d_1^2 P^\epsilon} (c_1 d_1^{-2})^{-N+n-1}$$

$$\ll P^{2(n-1)\epsilon} d_1^2 P^{-(N-n)\epsilon} \ll P^{-(N-3n)\epsilon}.$$

Here we allow for a possible renumbering of the variables. If $N = N(\epsilon)$ is chosen suitably, we get the lemma by combining this estimate with (2.10) and (2.20), on recalling that the summation over q is restricted to $q \ll P$.

Lemma 7. *Let* $|\boldsymbol{c}'| \leq P^\epsilon$. *Then*

$$\sum_{q=1}^{\infty} q^{-n} S_q(\boldsymbol{d}, \boldsymbol{c}) I_q(\boldsymbol{c}') = \eta(\boldsymbol{d}, \boldsymbol{c}) \sigma(\boldsymbol{d}, \boldsymbol{c}) \int_0^\infty q^{-n} I_q(\boldsymbol{c}') dq + O(P^{\alpha_n + 20\epsilon}). \tag{4.2}$$

Proof. Let

$$T(q) = \sum_{\ell \leq q} S_\ell(\boldsymbol{d}, \boldsymbol{c}),$$

$$B = \pi_{\boldsymbol{d}}^{3+\epsilon} (1 + |\boldsymbol{c}|)^{1/2}.$$

For $R \geq \frac{1}{2}$,

$$\sum_{R < q \leq 2R} q^{-n} S_q(\boldsymbol{d}, \boldsymbol{c}) I_q(\boldsymbol{c}') = \int_R^{2R} q^{-n} I_q(\boldsymbol{c}') dT(q) \tag{4.3}$$

$$= q^{-n} I_q(\boldsymbol{c}') T(q) \Big|_R^{2R} - \int_R^{2R} \frac{\partial}{\partial q} (q^{-n} I_q(\boldsymbol{c}')) T(q) dq$$

$$= q^{-n} I_q(\boldsymbol{c}') \left\{ \frac{\eta(\boldsymbol{d}, \boldsymbol{c}) \sigma(\boldsymbol{d}, \boldsymbol{c}) q^n}{n} + O(Bq^{\alpha_n + 2\epsilon}) \right\} \Bigg|_R^{2R}$$

$$- \int_R^{2R} \frac{\partial}{\partial q} (q^{-n} I_q(\boldsymbol{c}')) \left\{ \frac{\eta(\boldsymbol{d}) \sigma(\boldsymbol{d}, \boldsymbol{c}) q^n}{n} + O(Bq^{\alpha_n + 2\epsilon}) \right\} dq$$

from Lemma 4. Now for $R < q \leq 2R$,

$$q^{-n} I_q(\boldsymbol{c}') \ll P^{n/2+1+2\epsilon} R^{-n/2-1},$$

$$\frac{\partial}{\partial q} (q^{-n} I_q(\boldsymbol{c}')) \ll P^{n/2+1+2\epsilon} R^{-n/2-2}$$

from (2.12), (2.13). Hence the O-terms in the last expression in (4.3) contribute $O(BP^{n/2+1+2\epsilon}R^{-n/2-1+\alpha_n+2\epsilon})$. We conclude that

$$\sum_{R<q\leq 2R} q^{-n}S_q(\boldsymbol{d},\boldsymbol{c})I_q(\boldsymbol{c}') = \tag{4.4}$$

$$\eta(\boldsymbol{d},\boldsymbol{c})\sigma(\boldsymbol{d},\boldsymbol{c})\int_R^{2R} q^{-1}I_q(\boldsymbol{c}')dq + O(BP^{n/2+1+2\epsilon}R^{-n/2-1+\alpha_n+2\epsilon}).$$

The lemma follows because $q = O(P)$ for the nonzero terms of the series in (4.2).

Lemma 8. *We have*

$$\sum_{|\boldsymbol{c}'|>|\boldsymbol{d}|^\epsilon}\left|\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{c})I_q(\boldsymbol{c}')\right| \ll P^n.$$

Proof. By Lemma 6, we can restrict the sum to

$$|\boldsymbol{d}|^\epsilon < |\boldsymbol{c}'| \leq P^\epsilon.$$

Let $K > 1$. Combining Lemma 7 with (2.16), these $\boldsymbol{c}'$ contribute

$$\ll_K P^n \sum_{|\boldsymbol{c}'|>|\boldsymbol{d}|^\epsilon} |\boldsymbol{c}'|^{-K}|\sigma(\boldsymbol{d},\boldsymbol{c})| + P^{\alpha_n+24\epsilon}$$

$$\ll_K P^n \sum_{|\boldsymbol{c}'|>|\boldsymbol{d}|^\epsilon} |\boldsymbol{c}'|^{-K+\epsilon}\pi_{\boldsymbol{d}}^{2+\epsilon} + P^n$$

by (2.26), (2.28), (2.30) and Lemma 2 (i). The last expression is (arguing as in the proof of Lemma 6)

$$\ll_K P^{n+2n\epsilon}\pi_{\boldsymbol{d}}^{2+\epsilon}\sum_{c_1>d_1^2\,|\boldsymbol{d}|^\epsilon}(c_1d_1^{-2})^{-K+n-1+\epsilon} + P^n.$$

The lemma now follows from an application of (4.1) with $N = K-n-1-\epsilon$, $A = d_1^2$, $R = |\boldsymbol{d}|^\epsilon$; K is suitably chosen depending on ϵ.

Lemma 9. *Let*

$$0 < |\boldsymbol{c}'| \leq |\boldsymbol{d}|^\epsilon.$$

Then

$$\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{c})I_q(\boldsymbol{c}') \ll P^nA(\boldsymbol{d})\eta(\boldsymbol{d},\boldsymbol{c}) + P^{\alpha_n+20\epsilon}.$$

Proof. In view of Lemma 7 it suffices to show that

$$\sigma(\boldsymbol{d},\boldsymbol{c})\int_0^\infty q^{-n}I_q(\boldsymbol{c}')dq \ll P^nA(\boldsymbol{d}).$$

The integral is $\ll P^n\log(2\,|\boldsymbol{d}|)$ by (2.16), (2.17) and the simple observation that $|\boldsymbol{c}'| \geq |\boldsymbol{d}|^{-2}$. The required estimate for $\sigma(\boldsymbol{d},\boldsymbol{c})$ is provided by (2.26), (2.28), (2.30) and Lemma 2 (ii) (with $\epsilon/2$ in place of ϵ).

It remains to treat the series

$$\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})I_q(\boldsymbol{0}).$$

Lemma 10. *We have*

$$\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})I_q(\boldsymbol{d},\boldsymbol{0}) = e_n\sigma(\boldsymbol{d},\boldsymbol{0})\sigma_\infty(F,w)P^n\log P + O(P^nA(\boldsymbol{d})).$$

Proof. To begin with,

$$\sum_{q\le P^{1-\epsilon}} q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})I_q(\boldsymbol{0}) \tag{4.5}$$

$$= \sum_{q\le P^{1-\epsilon}} q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})P^n\sigma_\infty(F,w) + O_N(\pi_{\boldsymbol{d}}^2 P^{n+(1-\epsilon)/2}P^{-\epsilon N})$$

(from (2.14) and (2.20))

$$= e_n\sigma(\boldsymbol{d},\boldsymbol{0})\sigma_\infty(F,w)P^n\log P^{1-\epsilon} + O(P^nA(\boldsymbol{d}))$$

by Lemma 5 together with an appropriate choice of N.

For the range $q > P^{1-\epsilon}$, we use (4.4). Crudely,

$$\sum_{q>P^{1-\epsilon}} q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})I_q(\boldsymbol{c}') \tag{4.6}$$

$$= e_n\sigma(\boldsymbol{d},\boldsymbol{0})\int_{P^{1-\epsilon}}^\infty q^{-1}I_q(\boldsymbol{0})dq + O(\pi_{\boldsymbol{d}}^{3+\epsilon}P^{n/2+1+2\epsilon}).$$

Combining (4.5), (4.6), and substituting $I_q(\boldsymbol{0}) = P^nI_r^*(\boldsymbol{0})$, where $r = q/P$, we obtain

$$\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{0})I_q(\boldsymbol{0}) = e_n\sigma(\boldsymbol{d},\boldsymbol{0})\sigma_\infty(F,w)P^n\log P \tag{4.7}$$

$$+ e_n\sigma(\boldsymbol{d},\boldsymbol{0})L(P^{-\epsilon})P^n + O(P^nA(\boldsymbol{d})).$$

Here

$$L(\lambda) = \sigma_\infty(F,w)\log\lambda + \int_\lambda^\infty r^{-1}I_r^*(\boldsymbol{0})dr.$$

It is shown by Heath-Brown [4, p. 203] that $L(\lambda)$ tends to a limit $L(0)$ as λ tends to 0, and more precisely

$$L(\lambda) = L(0) + O_N(\lambda^N). \tag{4.8}$$

Recalling (2.28), (2.30) and Lemma 2 (ii), we see that (4.7) and (4.8) together yield the lemma.

Lemma 11. *We have*

$$N(F_{\boldsymbol{d}},w_{\boldsymbol{d}}) = e_n\pi_{\boldsymbol{d}}^{-2}\sigma(\boldsymbol{d},\boldsymbol{0})\sigma_\infty(F,w)P^{n-2}\log P$$

$$+ O(P^{n-2}\pi_{\boldsymbol{d}}^{-2}A(\boldsymbol{d})\#\{\boldsymbol{c} : |\boldsymbol{c}'| \le |\boldsymbol{d}|^\epsilon,\ \eta(\boldsymbol{d},\boldsymbol{c}) \ne 0\}).$$

Proof. Combining Lemmas 8, 9 and 10, and noting that $\boldsymbol{0}$ is counted in $\{\boldsymbol{c} : |\boldsymbol{c}'| \le |\boldsymbol{d}|^\epsilon$, $\eta(\boldsymbol{d},\boldsymbol{c}) \ne 0\}$,

$$\sum_{\boldsymbol{c}\in\mathbb{Z}^n}\sum_{q=1}^\infty q^{-n}S_q(\boldsymbol{d},\boldsymbol{c})I_q(\boldsymbol{d},\boldsymbol{c}')$$

$$= e_n\sigma(\boldsymbol{d},\boldsymbol{0})\sigma_\infty(F,w)P^n\log P$$

$$+ O(P^nA(\boldsymbol{d})\#\{\boldsymbol{c} : |\boldsymbol{c}'| \le |\boldsymbol{d}|^\epsilon,\ \eta(\boldsymbol{d},\boldsymbol{c}) \ne \boldsymbol{0}\}).$$

The lemma now follows easily on combining this with (1.5) and (1.3).

5. Completion of the proof of Theorems 1 and 2.

Lemma 12. *Suppose that F is nonsingular ($n = 4$) and robust ($n = 3$). In the notation of* (1.7)*, we have*

$$S_1(P^\epsilon) \ll P^n.$$

Proof. In view of (1.9), it suffices to show that

$$N(F_p, w_p) \ll P^{n+\epsilon/2} p^{-2}. \tag{5.1}$$

This is a consequence of [1, Proposition 1] for $n = 4$. The proof of that proposition can be adapted slightly to give (5.1) for $n = 3$. By following the argument on [1, pp. 107–8], we see that it suffices to show for $1 \le h \le P$ that the equation

$$cx_1^2 + z_1^2 + A_2 z_2^2 = 0$$

has $O(P^{1+\epsilon/2} h^{-1})$ solutions with

$$|(x, z_1, z_2)| \ll P, \quad x_1 \neq 0, \quad x_1 \equiv 0 \pmod{h}.$$

Here c, A_2 are *nonzero* integers, since the quadratic form $cx_1^2 + z_1^2 + A_2 z_2^2$ is obtained from F by a nonsingular linear change of variables. There are $O(Ph^{-1})$ choices for x_1. For each of these, there are $O(P^{\epsilon/2})$ possible (z_1, z_2) [1, Lemma 1]. This completes the proof of the lemma.

Lemma 13. *Let F be nonsingular. Let*

$$B(q) = \sum_{\boldsymbol{d} \mid q} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} S_q(\boldsymbol{d}, \boldsymbol{0}).$$

Then

(i) *$B(q)$ is a multiplicative function.*

(ii) $\displaystyle\sum_{\boldsymbol{t} \mid Q(z)} \frac{\mu(\boldsymbol{t})}{\pi_{\boldsymbol{t}}^2} S_q(\boldsymbol{t}, \boldsymbol{0}) = B(q) \prod_{\substack{p<z \\ p \nmid q}} (1 - p^{-2})^n.$

(iii) *For all primes p,*

$$1 + (1 - p^{-2})^{-n} \sum_{u=1}^{\infty} p^{-nu} B(p^u) = (1 - p^{-2})^{-n} \rho_p.$$

Proof. (i) This is a special case of [1, Lemma 17].

(ii) This is a variant of [1, Lemma 16]. The sum over $\boldsymbol{t}$ is unrestricted in [1].

(iii) This is obtained by letting N tend to infinity in the expression

$$1 + (1 - p^{-2})^{-n} \sum_{u=1}^{N} p^{-nu} B(p^u) = (1 - p^{-2})^{-n} p^{-(n-1)N} M_N,$$

where

$$M_N = \#\{\boldsymbol{x} \pmod{p^N} : F(\boldsymbol{x}) \equiv 0 \pmod{p^N}, p^2 \nmid x_1, \ldots, p^2 \nmid x_n\},$$

which is (5.9) of [1]. Convergence is a consequence of Lemma 3.

Lemma 14. *Let F be nonsingular ($n = 4$) and robust ($n = 3$). We have*

$$\sum_{\boldsymbol{d} \mid Q(z)} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} \sigma(\boldsymbol{d}, \boldsymbol{0}) = \rho^*(F) + O((\log\log P)^{-e_n}). \tag{5.2}$$

Proof. Let

$$V(w) = \prod_{p<w}\left(1 - \frac{1}{p}\right).$$

The left-hand side of (5.2) is $\lim_{w\to\infty} h(w)$, where

$$\begin{aligned} h(w) &= \sum_{\boldsymbol{d}\,|\,Q(z)} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} \prod_{p<w}\left(1 - \frac{1}{p}\right)\sum_{u=0}^{\infty} \frac{S_{p^u}(\boldsymbol{d}, \mathbf{0})}{p^{nu}} \\ &= V(w) \sum_{\substack{q=1 \\ p\,|\,q \Rightarrow p<w}}^{\infty} q^{-n} \sum_{\boldsymbol{d}\,|\,Q(z)} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} S_q(\boldsymbol{d}, \mathbf{0}) \end{aligned}$$

(after a simple manipulation). By Lemma 13 (ii),

$$\begin{aligned} h(w) &= V(w) \sum_{q=1}^{\infty} q^{-n} B(q) \prod_{\substack{p_1<z \\ p_1 \nmid q}} (1 - p_1^{-2})^n \\ &= V(w) \prod_{p_1<z} (1 - p_1^{-2})^n \sum_{\substack{q=1 \\ p\,|\,q \Rightarrow p<w}}^{\infty} C(q). \end{aligned}$$

Here

$$C(q) = q^{-n} B(q) \prod_{\substack{p_1<z \\ p_1\,|\,q}} (1 - p_1^{-2})^{-n}$$

is multiplicative by Lemma 13 (i), and so

$$\begin{aligned} h(w) &= V(w) \prod_{p<z} (1 - p_1^{-2})^n \prod_{p<w} (1 + C(p) + C(p^2) + \cdots) \\ &= \prod_{p_1<z} (1 - p_1^{-2})^n \prod_{p<w}\left(1 - \frac{1}{p}\right)\left(1 + a_p(z) \sum_{u=1}^{\infty} \frac{B(p^u)}{p^{nu}}\right). \end{aligned}$$

Here

$$a_p(z) = \begin{cases} (1 - p^{-2})^{-n} & \text{if } p < z \\ 1 & \text{if } p \geq z. \end{cases}$$

Letting w tend to infinity, the left-hand side of (5.2) is

$$\prod_{p<z}\left(1 - \frac{1}{p}\right)\rho_p \prod_{p\geq z}\left(1 - \frac{1}{p}\right)\left(1 + \sum_{u=1}^{\infty} \frac{B(p^u)}{p^{nu}}\right) \tag{5.3}$$

by Lemma 13 (iii). This is clearly close to $\rho^*(F)$ for large z. More precisely,

$$\left(1-\frac{1}{p}\right)\rho_p = \left(1-\frac{1}{p}\right)\Bigg((1-p^{-2})^{-n} + \sum_{u=1}^{\infty} p^{-nu} S_{p^u}(\mathbf{0})$$

$$+ \sum_{\substack{\boldsymbol{d}\mid p \\ \pi_{\boldsymbol{d}}>1}} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} \sum_{u\geq 1} \frac{S_{p^u}(\boldsymbol{d},\mathbf{0})}{p^{nu}}\Bigg)$$

$$= \left(1-\frac{1}{p}\right)\left(1 + \sum_{u=1}^{\infty} p^{-nu} S_{p^u}(\mathbf{0}) + O(p^{-(1+c_n)}\right)$$

by Lemma 3. Now for $p \nmid 2D$,

$$\left(1-\frac{1}{p}\right)\left(1+\sum_{u=1}^{\infty} p^{-nu} S_{p^u}(\mathbf{0})\right) = \begin{cases} 1+O(p^{-2}) & (n=4) \\ 1+O(p^{-3/2}) & (n=3) \end{cases}$$

as shown by Heath-Brown on p. 195 of [4]; one takes

$$\delta = \begin{cases} \frac{1}{6} & (n=3) \\ \frac{1}{2} & (n=4) \end{cases}$$

in his argument. We obtain

$$\left(1-\frac{1}{p}\right)\rho_p = 1 + O(p^{-(1+e_n)}). \tag{5.4}$$

Essentially the same argument shows that

$$\left(1-\frac{1}{p}\right)\left(1+\sum_{u=1}^{\infty} \frac{B(p^u)}{p^{nu}}\right) = 1 + O(p^{-(1+e_n)}). \tag{5.5}$$

It is now an easy matter to deduce from (5.4) and (5.5) that the expression in (5.3) is

$$\prod_p \left(1-\frac{1}{p}\right)\rho_p + O(z^{-e_n})$$

as required.

Proof of Proposition 1. Part (a) is a straightforward consequence of (5.4). For part (b), we may repeat verbatim the proof that $\rho_p > 0$ for all p in [1, pp. 130–131].

Proof of Proposition 2. We need only show that

$$\left(1-\frac{1}{p}\right)\rho_p = 1 - \frac{k}{p} + O\left(\frac{1}{p^{3/2}}\right)$$

where k is the number of j, $1 \leq j \leq 3$, for which $\det M_j = 0$. Arguing as in the preceding proof, this reduces to showing that

$$\sum_{\substack{\boldsymbol{d}\mid p \\ \pi_{\boldsymbol{d}}>1}} \frac{\mu(\boldsymbol{d})}{\pi_{\boldsymbol{d}}^2} \sum_{u\geq 1} \frac{S_{p^u}(\boldsymbol{d},\mathbf{0})}{p^{3u}} = -\frac{k}{p} + O\left(\frac{1}{p^{3/2}}\right). \tag{5.6}$$

Using unchanged the part of the proof of Lemma 3 with $\pi_{\boldsymbol{d}} \geq p^2$, we find that these terms contribute $O(p^{-3/2})$ to the left-hand side of (5.6). A familiar argument also gives, for $\pi_{\boldsymbol{d}} = p$,

$$\sum_{u \neq 2} \frac{S_p(\boldsymbol{d}, \boldsymbol{0})}{p^{3u}} \ll p^{-1/2}(d_1, p)(d_2, p)(d_3, p) + p^{-3/2}(d_1, p^2)(d_2, p^2)(d_3, p^2) \ll p^{1/2},$$

so that terms with $\pi_{\boldsymbol{d}} = p$, $u \neq 2$ also contribute $O(p^{-3/2})$.

Write $\boldsymbol{d}^{(1)} = (p, 1, 1)$, $\boldsymbol{d}^{(2)} = (1, p, 1)$, $\boldsymbol{d}^{(3)} = (1, 1, p)$ It remains to show that

$$\frac{S_{p^2}(\boldsymbol{d}^{(j)}, \boldsymbol{0})}{p^6} = \begin{cases} p + O(p^{1/2}) & \text{if } \det M_j = 0 \\ O(1) & \text{if } \det M_j \neq 0. \end{cases} \tag{5.7}$$

The case $\det M_j \neq 0$ of (5.7) is essentially the same as the case $n = 3$, $p \mid t_1$ of the proof of Lemma 2 (ii). Now suppose $\det M_j = 0$. Since M_j has rank at least 1, its rank is 1. Taking $j = 1$ for simplicity of writing,

$$\begin{bmatrix} x_2 & x_3 \end{bmatrix} M_j \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \frac{r}{s}(bx_2 + cx_3)^2$$

with integers r, s, b, c, $rs \neq 0$, $(b, c) \neq \boldsymbol{0}$. For $p \nmid s$, $s\bar{s} \equiv 1 \pmod{p^2}$,

$$F(p^2 x_1, x_2, x_3) \equiv r\bar{s}(bx_2 + cx_3)^2 \pmod{p^2}.$$

Hence

$$S_{p^2}(\boldsymbol{d}^{(j)}, \boldsymbol{0}) = p^2 \sideset{}{^*}\sum_{a=1}^{p^2} \sum_{x_2=1}^{p^2} \sum_{x_3=1}^{p^2} e_{p^2}(ar\bar{s}(bx_2 + cx_3)^2)$$

$$= p^4 \sideset{}{^*}\sum_{a=1}^{p^2} \sum_{y=1}^{p^2} e_p(ay^2) \quad \text{if } p \nmid rs(\gcd(b, c)),$$

since $bx_2 + cx_3$ takes each value $\pmod{p^2}$ exactly p^2 times. The last expression is evaluated in [4, Lemma 27] as

$$p^4 \cdot p^2(p-1) = p^7 - p^6,$$

and the proof of the proposition is complete.

Lemma 15. *Let F be nonsingular* ($n = 4$) *or robust* ($n = 3$)*. Then*

$$\sum_{\boldsymbol{d} \mid Q(z)} \mu(\boldsymbol{d}) N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}}) = e_n \sigma_\infty(F, w) \rho^*(F) P^{n-2} \log P + O(P^{n-2} \log P (\log\log P)^{-e_n}). \tag{5.8}$$

Proof. From Lemma 11, the left-hand side of (5.8) is

$$e_n \sigma_\infty(F, w) P^{n-2} \log P \sum_{\boldsymbol{d} \mid Q(z)} \frac{\mu(\boldsymbol{d}) \sigma(\boldsymbol{d}, \boldsymbol{0})}{\pi_{\boldsymbol{d}}^2} + O\left(P^{n-2} \sum_{\boldsymbol{d} \mid Q(z)} \pi_{\boldsymbol{d}}^{-2} A(\boldsymbol{d}) \#\{\boldsymbol{c} : |\boldsymbol{c}'| \leq |\boldsymbol{d}|^\epsilon\}\right). \tag{5.9}$$

The O-term is

$$\ll P^{n-2} \sum_{\boldsymbol{d} \mid Q(z)} \pi_{\boldsymbol{d}}^{-2+4/3+2+\epsilon} |\boldsymbol{d}|^{5\epsilon}$$

since $A(\boldsymbol{d}) = O(\pi_{\boldsymbol{d}}^{4/3+5\epsilon})$. Moreover, for given k,

$$\sum_{\boldsymbol{d} \mid Q(z)} \pi_{\boldsymbol{d}}^k \ll \#\{\boldsymbol{d} : \boldsymbol{d} \mid Q(z)\} Q(z)^{4k}$$
$$\ll Q(z)^{4k+\epsilon} \ll e^{z(4k+2\epsilon)}.$$

The O-term in (5.9) is thus

$$\ll P^{n-2} e^{6z} = P^{n-2} (\log P)^{6/7}.$$

The lemma now follows on applying Lemma 14 to the first sum over $\boldsymbol{d}$ in (5.9).

Lemma 16. *Under the hypothesis of Theorem 1 or Theorem 2, we have*

$$\sum_{z \le p < P^\epsilon} N(F_p, w_p) = O(P^{n-2} \log P (\log\log P)^{-1+8\epsilon}).$$

Proof. By Lemma 11,

$$N(F_p, w_p) = e_n p^{-2} \sigma(\boldsymbol{d}_p, 0) \sigma_\infty(F, w) P^{n-2} \log P + O(P^{n-2} p^{-2+5\epsilon} N_p). \tag{5.10}$$

Here N_p is the number of $\boldsymbol{c}$ in the box

$$\mathcal{B} : |c_1| \le p^{2+\epsilon},\ |c_j| \le p^\epsilon \quad (2 \le j \le n)$$

for which *either*

$$\det(M_{\boldsymbol{d}(p)}) M_{\boldsymbol{d}(p)}^{-1}(\boldsymbol{c}) = 0 \tag{5.11}$$

or $n = 3$ and

$$\det(M_{\boldsymbol{d}(p)}) M_{\boldsymbol{d}(p)}^{-1}(\boldsymbol{c}) = -q^2, \tag{5.12}$$

for a nonzero integer q.

Recalling (2.18), (2.19), we find that

$$\det(M_{\boldsymbol{d}(p)}) M_{\boldsymbol{d}(p)}^{-1}(\boldsymbol{c}) = b_{11} c_1^2 + 2p^2 \sum_{j=2}^n b_{1j} c_1 c_j + p^4 \sum_{i,j=2}^n b_{ij} c_i c_j,$$

with

$$b_{11} = \det(M_1) \ne 0.$$

We see at once that (5.11) holds for only $O(p^{3\epsilon})$ points $\boldsymbol{c}$ in $\mathcal{B}$, since $c_2, \ldots, c_n$ determine c_1 to within two choices.

If (5.12) holds, then

$$\begin{aligned} 0 &= b_{11} \det M_{\boldsymbol{d}(p)} M_{\boldsymbol{d}(p)}^{-1}(\boldsymbol{c}) + b_{11} q^2 \\ &= \left(b_{11} c_1 + p^2 \sum_{j=2}^n b_{1j} c_j \right)^2 - p^4 \ell + b_{11} q^2, \end{aligned} \tag{5.13}$$

where

$$\ell = \left(\sum_{j=2}^n b_{1j} c_j \right)^2 - \sum_{i,j=2}^n b_{ij} c_i c_j.$$

If $\ell = 0$, then $-b_{11}$ is a nonzero square from (5.13), in contradiction to the hypothesis of Theorem 2. We conclude that the aid of [1, Lemma 1] that for given c_2, c_3, (5.12) determines c_1 to within $O(p^\epsilon)$ possibilities. Thus in all cases,

$$N_p = O(p^{3\epsilon}).$$

We use this estimate together with (2.28), (2.30) and Lemma 2 (ii) to deduce from (5.10) that

$$N(F_p, w_p) = O(P^{n-2}(\log P)p^{-2+8\epsilon}).$$

The lemma now follows.

We are now ready to complete the proofs of Theorems 1 and 2. From (1.6), (1.8),

$$\left| R(F,w) - \sum_{\boldsymbol{d} \mid Q(z)} \mu(\boldsymbol{d}) N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}}) \right| \le n \max_j S_j(z) \le n \sum_{p \ge z} N(F_p, w_p)$$

after a possible renumbering of the variables. Thus

$$R(F,w) = \sum_{\boldsymbol{d} \mid Q(z)} \mu(\boldsymbol{d}) N(F_{\boldsymbol{d}}, w_{\boldsymbol{d}}) + O(P^{n-2}(\log P)(\log\log P)^{-1+8\epsilon})$$

(from Lemmas 12 and 16)

$$= e_n \sigma_\infty(F,w)\rho^*(F)P^{n-2}\log P + O(P^{n-2}(\log P)(\log\log P)^{-g_n})$$

from Lemma 15. Here

$$g_n = \begin{cases} 1 - 8\epsilon & (n = 3) \\ 1/2 & (n = 4). \end{cases}$$

Since ϵ is arbitrary, this completes the proof.

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