Schäffer's determinant argument

R. C. Baker

1 Introduction

Let $\|\ldots\|$ denote distance from the nearest integer. Various versions of the following problem in simultaneous Diophantine approximation have been studied since 1957, beginning with Danicic [5]. Given an integer $h \geq 2$. we seek a number θ having the following property, for every $\epsilon > 0$ and every pair $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_h), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_h)$ in $\mathbb{R}^h$:

For $N > C(h, \epsilon)$, there is an integer $n, 1 \leq n \leq N$, satisfying

$$\|n^2 \alpha_j + n\beta_j\| < N^{-\theta+\epsilon} \quad (j = 1, \ldots, h).$$

It is convenient to say that θ is *admissible for h quadratic polynomials* if θ possesses the above property. The best known result for general h is that

$$\frac{1}{h^2+h} \text{ is admissible for } h \text{ quadratic polynomials.} \tag{1.1}$$

Most of the ideas leading to (1.1) occur in the lectures of W. M. Schmidt [7]. In particular [7] contains the corresponding result for the special case $\boldsymbol{\beta} = \mathbf{0}$. The finishing touches for (1.1) are in Baker [1], [2]; see also [3]. One should note the correction in [4], which applies equally to Theorem 5.1 of [3]. This theorem is used in proving (1.1) in [3], and again in the present paper.

Schäffer [6] was able to improve (1.1) in the case $h = 2$, showing that $2/11$ is admissible for a pair of quadratic polynomials. The key to his improvement is Lemma 4 of [6], which we need not restate here since it is essentially subsumed under Theorems 2 and 3 below. Schäffer's lemma is an ingenious refinement of the 'determinant argument' of Schmidt. This is Lemma 18A of [7], abstracted as Lemma 7.6 in [3] and repeated below as Lemma 4.

Theorems 2 and 3 will be applied to give the following modest improvement of (1.1).

Theorem 1 *Let $h \geq 3$. The number $(h^2 + h - 1/2)^{-1}$ is admissible for h quadratic polynomials.*

We now give a version of Schäffer's lemma for $\mathbb{R}^h$. We write $\boldsymbol{ab}$ for inner product in $\mathbb{R}^h$, and $|\boldsymbol{a}| = (\boldsymbol{aa})^{1/2}$. The constants $C(h, \epsilon), C(h)$ need not be the same at each occurrence. The cardinality of a finite set $\mathcal{E}$ is denoted by $|\mathcal{E}|$.

Theorem 2 *Let $h \geq 2$, $\epsilon > 0$, $M > C(h, \epsilon)$, $A \geq 1$, $U \geq 1$, $UA \leq M$ and $0 < V < 1$, with*

$$M^{h-1+\epsilon} AV < 1. \tag{1.2}$$

Let $\boldsymbol{e} \in \mathbb{R}^h$. Let $\mathcal{A}$ be a subset of $\mathbb{Z}^h$, with

$$|\mathcal{A}| > M^{2\epsilon} \max(1, (M^h V)^{h/(h+1)}).$$

Suppose that, for $\boldsymbol{p}$ in $\mathcal{A}$, we have

$$|\boldsymbol{p}| \leq A, \tag{1.3}$$

and there are coprime integers $\ell(\boldsymbol{p}), w(\boldsymbol{p})$,

$$0 < \ell(\boldsymbol{p}) \leq U, \tag{1.4}$$

with

$$|\ell(\boldsymbol{p})\boldsymbol{pe} - w(\boldsymbol{p})| < V. \tag{1.5}$$

Then there is a subset $\mathcal{C}$ of $\mathcal{A}$ and a natural number ℓ such that

$$|\mathcal{C}| \geq |\mathcal{A}| M^{-\epsilon} \min(1, (M^h V)^{-h/(h+1)})$$

and $\ell(\boldsymbol{p}) = \ell$ for all $\boldsymbol{p}$ in $\mathcal{C}$.

In Theorem 3, we assume a somewhat similar situation but we suppose that there is some 'known repetition' among the $\ell(\boldsymbol{p})$. We use this to get a 'lot of repetition'. The linear span of a set S in $\mathbb{R}^h$ is denoted by $\operatorname{Span} S$.

Theorem 3 *Let $h \geq 2$, $\epsilon > 0$, $M > C(h, \epsilon)$, $A \geq 1, U \geq 1$, $UA \leq M$, $0 < V < 1$ and let $\boldsymbol{e} \in \mathbb{R}^h$. Let $\mathcal{A}$ be a subset of $\mathbb{Z}^h, W = \operatorname{Span} \mathcal{A}$, $\dim W = m$. Suppose that, for each $\boldsymbol{p}$ in $\mathcal{A}$,*

$$A/2 < |\boldsymbol{p}| \leq A, \tag{1.6}$$

and there exist coprime integers $\ell(\boldsymbol{p}), w(\boldsymbol{p})$ *satisfying*

$$U/2 < \ell(\boldsymbol{p}) \leq U, \tag{1.7}$$

$$|\ell(\boldsymbol{p})\boldsymbol{p}\boldsymbol{e} - w(\boldsymbol{p})| < V. \tag{1.8}$$

Suppose that for some integer n, $2 \leq n \leq m$ *with*

$$C(h)U^{1+m-n}A^m V < 1 \tag{1.9}$$

for a suitable positive $C(h)$, *there are linearly independent* $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ *in* $\mathcal{A}$ *with* $\ell(\boldsymbol{p}_1) = \ell(\boldsymbol{p}_2) = \cdots = \ell(\boldsymbol{p}_n)$. *Then there is a subset* $\mathcal{C}$ *of* $\mathcal{A}$ *and a natural number* ℓ' *such that*

$$|\mathcal{C}| > |\mathcal{A}|M^{-\epsilon}$$

and $\ell(\boldsymbol{p}) = \ell'$ *for all* $\boldsymbol{p}$ *in* $\mathcal{C}$.

Lemma 4 of [6] is essentially equivalent to the cases $h = 2$ of Theorems 3 and 4, taken together.

2 Proofs of Theorems 2 and 3.

As in [3], the *determinant* of t vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t$ in $\mathbb{R}^h$, where $1 \leq t \leq h$, is the t-dimensional volume of the parallellepiped

$$\left\{ \sum_{i=1}^{t} y_i \boldsymbol{a}_i : 0 \leq y_1, \ldots, y_t \leq 1 \right\}$$

and is denoted by $\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t)$. Note that

$$\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t)^2 = \det\{\boldsymbol{a}_i \boldsymbol{a}_j : 1 \leq i, j \leq t\}$$

is an integer whenever $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t$ are in $\mathbb{Z}^h$; compare [8], equation (2.1), p. 4. If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t$ are linearly independent, and

$$\Lambda = \left\{ \sum_{i=1}^{t} n_i \boldsymbol{a}_i : n_1, \ldots, n_t \in \mathbb{Z} \right\}$$

is the t-dimensional lattice generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t$, then the *determinant* of Λ is defined to be

$$d(\Lambda) = \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t).$$

The unit ball in $\mathbb{R}^h$ is denoted by K_0.

We begin with a few observations from linear algebra.

Lemma 1 *Let* $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_h$ *be in* $\mathbb{R}^h, \boldsymbol{v}_0 \neq \mathbf{0}$. *Then*

$$\det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_h) \leq h \max\left(\frac{|\boldsymbol{v}_1|}{|\boldsymbol{v}_0|}, \dots, \frac{|\boldsymbol{v}_h|}{|\boldsymbol{v}_0|}\right) \max_i \det(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_h).$$

Proof. Evidently we may suppose that $|\boldsymbol{v}_0| = 1$ and, after applying a linear isometry to $\mathbb{R}^h$, that $\boldsymbol{v}_0 = (1, 0, \dots, 0)$. Let $\boldsymbol{v}_i = (v_{i1}, \dots, v_{ih})$, and let M_i be the cofactor of v_{i1} in the matrix $A = [v_{ij} : 1 \leq i, j \leq h]$. Then

$$\det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_h) \leq \sum_{i=1}^{h} |v_{i1} M_i| \leq h \max_i |\boldsymbol{v}_i| \max_i |M_i|. \tag{2.1}$$

Now consider the matrix A_i obtained by replacing row i of A by $\boldsymbol{v}_0$. We have

$$\det(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_h) = |\det A_i| = |M_i|. \tag{2.2}$$

The lemma follows from (2.1), (2.2).

Lemma 2 *Let* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_h$ *be linearly independent in* $\mathbb{R}^h$. *The distance between parallel hyperplanes*

$$\boldsymbol{c} + a_i \boldsymbol{x}_h + \text{Span}\,\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1}\} \quad (i = 1, 2)$$

is

$$|a_1 - a_2| \frac{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1}, \boldsymbol{x}_h)}{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1})}.$$

Proof. It suffices to show that the distance d from $\boldsymbol{x}_h$ to $\text{Span}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1}\}$ is

$$\frac{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_h)}{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1})}.$$

We use the Gram-Schmidt process to replace $\boldsymbol{x}_1, \dots, \boldsymbol{x}_h$ by an orthogonal set

$$\boldsymbol{v}_1 = \boldsymbol{x}_1, \boldsymbol{v}_2 = \boldsymbol{x}_2 - \frac{\boldsymbol{x}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1$$

and so on. Note that

$$\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_i) = \det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i) = |\boldsymbol{v}_1| \dots |\boldsymbol{v}_i| \quad (i = 1, \dots, h).$$

Hence

$$d = |\boldsymbol{v}_h| = \frac{\det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_h)}{\det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{h-1})} = \frac{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_h)}{\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h-1})}.$$

Lemma 3 *Let* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_h$ *be linearly independent points of a hyperplane with equation*

$$\boldsymbol{a}\boldsymbol{x} = b$$

in $\mathbb{R}^h$*. Then for any* $\boldsymbol{x}$ *in the hyperplane, we have*

$$\boldsymbol{x} = t_1\boldsymbol{x}_1 + \cdots + t_h\boldsymbol{x}_h \tag{2.3}$$

with $t_1 + \cdots + t_h = 1$*.*

Proof. We may define $t_1, \ldots, t_h$ uniquely via (2.3). On each side of (2.3), take the inner product with $\boldsymbol{a}$:

$$\begin{aligned} b = \boldsymbol{a}\boldsymbol{x} &= t_1\boldsymbol{a}\boldsymbol{x}_1 + \cdots + t_h\boldsymbol{a}\boldsymbol{x}_h \\ &= (t_1 + \cdots + t_h)b. \end{aligned}$$

Since $b \neq 0$ by independence of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_h$, the lemma follows.

Throughout the remainder of the paper, constants implied by $\ll$ depend at most on h and ϵ. We suppose (as we may) that ϵ is sufficiently small. A quantity of the form $C(h)\epsilon^2$ is denoted by δ.

Lemma 4 *Let* $N > C(h, \epsilon)$*. Let* Λ *be an* h*-dimensional lattice in* $\mathbb{R}^h$ *with* $d(\Lambda) = D \le N^2$ *and* $\Lambda \cap K_0 = \{\mathbf{0}\}$*. Let* Π *be the dual lattice of* Λ*. Let* $\mathcal{A}$ *be a subset of* Π *with* $|\boldsymbol{p}| \le N$ *for all* $\boldsymbol{p}$ *in* $\mathcal{A}$*. Suppose that* Span $\mathcal{A}$ *has dimension* t*, and that any* t *vectors in* $\mathcal{A}$ *have determinant* $\le Z$*. Let* $\boldsymbol{e} \in \mathbb{R}^h$*. Let* U, V *be positive numbers,* $U \le N$*, such that for any* $\boldsymbol{p}$ *in* $\mathcal{A}$ *there are coprime integers* $\ell(\boldsymbol{p}), w(\boldsymbol{p})$ *satisfying*

$$1 \le \ell(\boldsymbol{p}) \le U, \ |\ell(\boldsymbol{p})\boldsymbol{e}\boldsymbol{p} - w(\boldsymbol{p})| < V.$$

Suppose further that

$$ZU^tVDN^\epsilon \le 1.$$

Then there is an integer ℓ *and a subset* $\mathcal{C}$ *of* $\mathcal{A}$ *with* $|\mathcal{C}| \ge |\mathcal{A}|N^{-\delta}, \ell(\boldsymbol{p}) = \ell$ *for all* $\boldsymbol{p}$ *in* $\mathcal{C}$*.*

Proof. As noted above, this is Lemma 7.6 of [3].

Proof of Theorem 2. There are two cases to consider.

Case 1. There is a subset $\mathcal{E}$ of $\mathcal{A}$ with

$$|\mathcal{E}| > |\mathcal{A}|M^{-\epsilon/2}\min(1, (M^hV)^{-h/(h+1)}),$$

such that Span $\mathcal{E}$ has dimension $t \le h-1$.

We apply Lemma 4 with $\Lambda = \Pi = \mathbb{Z}^h$, $D = 1$, $N = M$ and $\mathcal{E}$, ϵ^2 in place of $\mathcal{A}$, ϵ. Clearly we may take

$$Z = A^t.$$

Now

$$\begin{aligned} ZU^tVDN^{\epsilon^2} &\ll A^tU^tVM^{\epsilon^2} \\ &\ll M^{h-1+\epsilon^2}V \ll M^{-\epsilon^2}. \end{aligned}$$

Hence there is a subset $\mathcal{C}$ of $\mathcal{E}$ and a natural number ℓ such that

$$|\mathcal{C}| \ge |\mathcal{E}|M^{-\delta} > |\mathcal{A}|M^{-\epsilon}\min(1, (M^hV)^{-h/(h+1)})$$

and $\ell(\boldsymbol{p}) = \ell$ for all $\boldsymbol{p}$ in $\mathcal{C}$.

Case 2. Case 1 does not hold.

It is convenient to write

$$f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{e}, R = (M/V)^{1/(h+1)}.$$

By Dirichlet's theorem, there is a point $\boldsymbol{p}_0$ in $\mathbb{Z}^h$ and an integer w_0 such that

$$0 < |\boldsymbol{p}_0| \le R, \ |f(\boldsymbol{p}_0) - w_0| \ll R^{-h}. \tag{2.4}$$

We choose $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{h-1}$ in $\mathcal{A}$ to maximize

$$C = \det(\boldsymbol{p}_0, \ell(\boldsymbol{p}_1)\boldsymbol{p}_1, \ldots, \ell(\boldsymbol{p}_{h-1})\boldsymbol{p}_{h-1}).$$

Since we are in Case 2, we have Span $\mathcal{A} = \mathbb{R}^h$ and $C > 0$. Let us write

$$\ell_j = \ell(\boldsymbol{p}_j), w_j = w(\boldsymbol{p}_j) \quad (j = 1, \ldots, h).$$

We note that

$$\det(\boldsymbol{p}_0, \ell_1\boldsymbol{p}_1, \ldots, \ell_{j-1}\boldsymbol{p}_{j-1}, \ell(\boldsymbol{p})\boldsymbol{p}, \ell_{j+1}\boldsymbol{p}_{j+1}, \ldots, \ell_{h-1}\boldsymbol{p}_{h-1}) \le C \tag{2.5}$$

for all $\boldsymbol{p}$ in $\mathcal{A}$, by choice of $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{h-1}$, while

$$\det(\ell(\boldsymbol{p})\boldsymbol{p}, \ell_1\boldsymbol{p}_1, \ldots, \ell_{h-1}\boldsymbol{p}_{h-1}) \ll \frac{M}{|\boldsymbol{p}_0|}C \tag{2.6}$$

by Lemma 1, (2.5), (1.3) and (1.4).

It follows from (2.5), (2.6) and Cramer's rule that if we write $\ell(\boldsymbol{p})\boldsymbol{p}$ in the form

$$\ell(\boldsymbol{p})\boldsymbol{p} = y_0\boldsymbol{p}_0 + y_1\boldsymbol{p}_1 + \cdots + y_{h-1}\boldsymbol{p}_{h-1}, \tag{2.7}$$

then

$$|y_i| \le 1 \quad (i = 1, \ldots, h-1), \ |y_0| \ll \frac{M}{|\boldsymbol{p}_0|}. \tag{2.8}$$

Let $E(\boldsymbol{x})$ be the linear function on $\mathbb{R}^h$ for which

$$E(\boldsymbol{p}_0) = w_0, \ E(\ell_j\boldsymbol{p}_j) = w_j \quad (j = 1, \ldots, h-1).$$

Then E takes the form

$$E(\boldsymbol{x}) = \frac{1}{C}\boldsymbol{B}\boldsymbol{x}$$

with $\boldsymbol{B} = (B_1, \ldots, B_h) \in \mathbb{Z}^h$. Let us write

$$\gcd\,(B_1, \ldots, B_h, C) = D$$

for the greatest common divisor of $B_1, \ldots, B_h$ and C.

Now consider the linear function

$$F = f - E.$$

We have

$$F(\boldsymbol{p}_0) = f(\boldsymbol{p}_0) - w_0 \ll R^{-h},$$
$$F(\ell_j\boldsymbol{p}_j) = f(\ell_j\boldsymbol{p}_j) - w_j \ll V \quad (j = 1, \ldots, h-1),$$

from (2.4), (1.5). Taking into account (2.7), (2.8),

$$\begin{aligned} F(\ell(\boldsymbol{p})\boldsymbol{p}) &\ll \frac{M}{|\boldsymbol{p}_0|} R^{-h} + V \\ &\ll |\boldsymbol{p}_0|^{-1}(MR^{-h} + RV) \\ &\ll |\boldsymbol{p}_0|^{-1}M^{1/(h+1)}V^{h/(h+1)} \end{aligned} \tag{2.9}$$

for all $\boldsymbol{p}$ in $\mathcal{A}$. It is convenient to define

$$H = |\boldsymbol{p}_0|^{-1}M^{1/(h+1)}V^{h/(h+1)};$$

the above calculation gives $V \ll H$.

We can now give a bound for the integer

$$k(\boldsymbol{p}) = CD^{-1}(E(\ell(\boldsymbol{p})\boldsymbol{p}) - w(\boldsymbol{p})). \tag{2.10}$$

We have

$$\begin{aligned} k(\boldsymbol{p}) &= CD^{-1}(-F(\ell(\boldsymbol{p})\boldsymbol{p}) + f(\ell(\boldsymbol{p})\boldsymbol{p}) - w(\boldsymbol{p})) \\ &\ll CD^{-1}(H+V) \ll CD^{-1}H \end{aligned}$$

from (2.9), (1.5).

Next we distinguish two subcases of Case 2.

Case 2(a). $CD^{-1}H < M^{-\epsilon^2}$. In this case $k(\boldsymbol{p}) = 0$ for all $\boldsymbol{p}$ in $\mathcal{A}$. The points

$$(\ell(\boldsymbol{p})\boldsymbol{p}, w(\boldsymbol{p}))$$

lie in an h-dimensional hyperplane in $\mathbb{R}^{h+1}$. If we fix any h linearly independent points $\boldsymbol{p}_1', \ldots, \boldsymbol{p}_h'$ of $\mathcal{A}$, then for any $\boldsymbol{p}$ in $\mathcal{A}$,

$$\det \begin{bmatrix} \ell(\boldsymbol{p})\boldsymbol{p} & w(\boldsymbol{p}) \\ \ell(\boldsymbol{p}_1')\boldsymbol{p}_1' & w(\boldsymbol{p}_1') \\ \vdots & \\ \ell(\boldsymbol{p}_h')\boldsymbol{p}_h' & w(\boldsymbol{p}_h') \end{bmatrix} = 0.$$

Expanding by the first row,

$$0 = \ell(\boldsymbol{p})G \pm w(\boldsymbol{p}) \det(\ell(\boldsymbol{p}_1')\boldsymbol{p}_1', \ldots, \ell(\boldsymbol{p}_h')\boldsymbol{p}_h')$$

for some integer G, so that $\ell(\boldsymbol{p})$ is a divisor of

$$L = \det(\ell(\boldsymbol{p}_1')\boldsymbol{p}_1', \ldots, \ell(\boldsymbol{p}_h')\boldsymbol{p}_h').$$

Since $L \leq M^h, L$ has at most M^ϵ divisors, and there is a divisor ℓ of L such that

$$\ell(\boldsymbol{p}) = \ell$$

for $\boldsymbol{p}$ in a subset $\mathcal{F}$ of $\mathcal{A}$ with

$$|\mathcal{F}| > |\mathcal{A}|M^{-\epsilon}.$$

Case 2(b). $CD^{-1}H \geq M^{-\epsilon^2}$. In this case, there are

$$\ll CD^{-1}H + 1 \ll CD^{-1}HM^{\epsilon^2}$$

possible values of $k(\boldsymbol{p})$. There is an integer k and a subset $\mathcal{A}_1$ of $\mathcal{A}$ with

$$|\mathcal{A}_1| \gg |\mathcal{A}|C^{-1}DH^{-1}M^{-\epsilon^2}, \tag{2.11}$$

$$k(\boldsymbol{p}) = k \text{ for all } \boldsymbol{p} \text{ in } \mathcal{A}_1. \tag{2.12}$$

In particular, the subset S of $\mathbb{Z}^h$ consisting of solutions of

$$D^{-1}\boldsymbol{Bx} \equiv k \pmod{CD^{-1}}$$

contains $\{\ell(\boldsymbol{p})\boldsymbol{p} : \boldsymbol{p} \in \mathcal{A}_1\}$. Now S is a translate $\Lambda_0 + \boldsymbol{R}$ of the sublattice Λ_0 of $\mathbb{Z}^h$ consisting of solutions of

$$D^{-1}\boldsymbol{Bx} \equiv 0 \pmod{CD^{-1}}.$$

It is easy to see that $\det \Lambda_0 = CD^{-1}$.

The lattice Λ_1 generated by $\boldsymbol{p}_0, \ell_1\boldsymbol{p}, \ldots, \ell_{h-1}\boldsymbol{p}_{h-1}$ is contained in Λ_0, since

$$D^{-1}\boldsymbol{Bp}_0 = CD^{-1}w_0, D^{-1}B\ell_j\boldsymbol{p}_j = CD^{-1}w_j \quad (j = 1, \ldots, h-1).$$

The index of Λ_1 in Λ_0 is

$$\frac{\det \Lambda_1}{\det \Lambda_0} = \frac{C}{CD^{-1}} = D.$$

Hence we can write Λ_0 as a union of D translates of Λ_1. We conclude that there is a $\boldsymbol{Q}$ in $\mathbb{Z}^h$ and a subset $\mathcal{A}_2$ of $\mathcal{A}_1$ such that

$$\ell(\boldsymbol{p})\boldsymbol{p} \in \boldsymbol{Q} + \Lambda_1 \quad (\boldsymbol{p} \in \mathcal{A}_2), \tag{2.13}$$

$$|\mathcal{A}_2| \geq |\mathcal{A}_1|D^{-1} \gg |\mathcal{A}|C^{-1}H^{-1}M^{-\epsilon^2}, \tag{2.14}$$

from (2.11).

We now seek a hyperplane that contains many of the points $\ell(\boldsymbol{p})\boldsymbol{p}$ with $\boldsymbol{p}$ in $\mathcal{A}_2$. For $n \in \mathbb{Z}$, let

$$L_n = \boldsymbol{Q} + n\ell_{h-1}\boldsymbol{p}_{h-1} + \text{Span}\,\{\boldsymbol{p}_0, \ell_1\boldsymbol{p}_1, \ldots, \ell_{h-2}\boldsymbol{p}_{h-2}\}.$$

If n_0 and n_1 are the smallest and largest integers for which L_n meets the ball MK_0, then

$$n_0 - n_1 \ll \frac{M \det(\boldsymbol{p}_0, \ell_1 \boldsymbol{p}_1, \ldots, \ell_{h-2}\boldsymbol{p}_{h-2})}{C}$$

(by Lemma 2)

$$\ll \frac{M^{h-1}|\boldsymbol{p}_0|}{C}.$$

Since $C \le |\boldsymbol{p}_0| M^{h-1}$, it follows that there is an n for which

$$\begin{aligned} |\{\boldsymbol{p} \in \mathcal{A}_2 : \ell(\boldsymbol{p})\boldsymbol{p} \in L_n\}| &\gg |\mathcal{A}_2| C M^{-h+1} |\boldsymbol{p}_0|^{-1} \\ &\gg |\mathcal{A}| H^{-1} M^{-h+1-\delta} |\boldsymbol{p}_0|^{-1} \end{aligned}$$

(from (2.14))

$$\gg |\mathcal{A}| V^{-h/(h+1)} M^{-h^2/(h+1)-\delta}$$

from the definition of H. Let

$$\mathcal{A}_3 = \{\boldsymbol{p} \in \mathcal{A}_2 : \ell(\boldsymbol{p})\boldsymbol{p} \in L_n\}.$$

Since we are in Case 2, $\operatorname{Span} \mathcal{A}_3$ is $\mathbb{R}^h$. We select linearly independent points $\boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_h$ in $\mathcal{A}_3$.

Recalling Lemma 3, for any $\boldsymbol{p}$ in $\mathcal{A}_3$, there are real $t_1, \ldots, t_h$ with

$$t_1 + \cdots + t_h = 1, \tag{2.15}$$

$$\ell(\boldsymbol{p})\boldsymbol{p} = t_1 \ell(\boldsymbol{p}'_1)\boldsymbol{p}'_1 + \cdots + t_h \ell(\boldsymbol{p}'_h)\boldsymbol{p}'_h. \tag{2.16}$$

Now

$$\det \begin{bmatrix} \ell(\boldsymbol{p})\boldsymbol{p} & w(\boldsymbol{p}) \\ \ell(\boldsymbol{p}'_1)\boldsymbol{p}'_1 & w(\boldsymbol{p}'_1) \\ \vdots & \\ \ell(\boldsymbol{p}'_h)\boldsymbol{p}'_h & w(\boldsymbol{p}'_h) \end{bmatrix} = \det \begin{bmatrix} \ell(\boldsymbol{p})\boldsymbol{p} & -kC^{-1}D \\ \ell(\boldsymbol{p}'_1)\boldsymbol{p}'_1 & -kC^{-1}D \\ \vdots & \\ \ell(\boldsymbol{p}'_h)\boldsymbol{p}'_h & -kC^{-1}D \end{bmatrix}$$

(subtract B_j/C times column j from column $h+1$ for $j=1,\dots,h$ and use (2.10), (2.12))

$$= \det \begin{bmatrix} \mathbf{0} & 0 \\ \ell(\boldsymbol{p}_1')\boldsymbol{p}_1' & -kC^{-1}D \\ \vdots & \\ \ell(\boldsymbol{p}_h')\boldsymbol{p}_h' & -kC^{-1}D \end{bmatrix} = 0.$$

For the penultimate step, we subtract t_1 times row $2,\dots,t_h$ times row $h+1$ from row 1 and use (2.15), (2.16).

We can now argue as in Case 2 (a) to show that there is a subset $\mathcal{C}$ of $\mathcal{A}_3$, on which $\ell(\boldsymbol{p})$ is constant, say $\ell(\boldsymbol{p}) = \ell'$, satisfying

$$|\mathcal{C}| \geq |\mathcal{A}_3| M^{-\delta} \geq |\mathcal{A}| M^{-\epsilon} (M^h V)^{-h/(h+1)}.$$

Thus a subset $\mathcal{C}$ of $\mathcal{A}$ with the required properties exists in all cases.

Proof of Theorem 3. Let Γ denote the m-dimensional lattice $\mathbb{Z}^h \cap W$; let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ be a basis of Γ. Let $\boldsymbol{p}_{n+1}, \dots, \boldsymbol{p}_m$ be chosen in $\mathcal{A}$ so that $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m$ is a basis of W. Let us write $\ell(\boldsymbol{p}_j) = \ell_j$, $w(\boldsymbol{p}_j) = w_j$ $(j = 1, \dots, m)$. Thus

$$\ell_j = \ell_1 \qquad (j = 2, \dots, n). \tag{2.17}$$

We now write

$$\boldsymbol{p}_j = p_{j1}\boldsymbol{x}_1 + \cdots + p_{jm}\boldsymbol{x}_m \quad (j = 1, \dots, m),$$

so that the p_{jk} are integers. Let P be the matrix $[\ell_j p_{jk}]_{1 \leq j,k \leq m}$. Then

$$\det(\ell_1 \boldsymbol{p}_1, \dots, \ell_m \boldsymbol{p}_m) = |\det P| \det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m). \tag{2.18}$$

We now imitate the construction in the previous proof. Define the linear function E_1 on W by the conditions

$$E_1(\ell_j \boldsymbol{p}_j) = w_j \qquad (j = 1, \dots, m).$$

Let $A_j = E_1(\boldsymbol{x}_j)$, so that

$$E_1(\alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_m \boldsymbol{x}_m) = A_1\alpha_1 + \cdots + A_m\alpha_m.$$

Since

$$E_1(\ell_j p_{j1}\boldsymbol{x}_1 + \cdots + \ell_j p_{jm}\boldsymbol{x}_m) = E_1(\ell_j \boldsymbol{p}_j) = w_j,$$

we have

$$A_1\ell_j p_{j1} + \cdots + A_m \ell_j p_{jm} = w_j \quad (j = 1, \ldots, m).$$

If we solve for A_i by Cramer's rule, we obtain

$$|A_i| = \frac{\det P_i}{\det P}, \tag{2.19}$$

where P_i is obtained from P by replacing column i by a column with entries $w_1, \ldots, w_m$. Clearly we may cancel ℓ_1^{n-1} from numerator and denominator on the right side of (2.19). This gives

$$A_i = \frac{B_i}{\ell_1^{-n+1} \det P} \qquad (B_i \in \mathbb{Z}),$$

so that

$$\ell_1^{-n+1} (\det P) E_1(\boldsymbol{p}) \in \mathbb{Z} \quad (\boldsymbol{p} \in \mathcal{A}). \tag{2.20}$$

We observe that

$$\begin{aligned} |\det P| &= \frac{\det(\ell_1 \boldsymbol{p}_1, \ldots, \ell_m \boldsymbol{p}_m)}{\det(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)} \\ &\ll \det(\ell_1 \boldsymbol{p}_1, \ldots, \ell_m \boldsymbol{p}_m) \end{aligned} \tag{2.21}$$

from (2.18).

Now let $F_1 = f - E_1$. If we write $\ell(\boldsymbol{p})\boldsymbol{p}$ in the form

$$\ell(\boldsymbol{p})\boldsymbol{p} = \alpha_1 \ell_1 \boldsymbol{p}_1 + \cdots + \alpha_m \ell_m \boldsymbol{p}_m,$$

then

$$\begin{aligned} |\alpha_i| &= \frac{\det(\ell_1 \boldsymbol{p}_1, \ldots, \ell_{i-1} \boldsymbol{p}_{i-1}, \ell(\boldsymbol{p})\boldsymbol{p}, \ell_{i+1} \boldsymbol{p}_{i+1}, \ldots, \ell_m \boldsymbol{p}_m)}{\det(\ell_1 \boldsymbol{p}_1, \ldots, \ell_m \boldsymbol{p}_m)} \\ &\ll \frac{A^m}{\det(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)} \end{aligned}$$

by (1.6), (1.7). Hence

$$\begin{aligned} F_1(\ell(\boldsymbol{p})\boldsymbol{p}) &\ll \frac{A^m}{\det(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)} \max_i |F_1(\ell_i \boldsymbol{p}_i)| \\ &\ll \frac{A^m}{\det(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)} V \end{aligned} \tag{2.22}$$

for all $\boldsymbol{p}$ in $\mathcal{A}$, by (1.8) and the definition of F_1.

Given $\boldsymbol{p}$ in $\mathcal{A}$, we now estimate the integer

$$k(\boldsymbol{p}) = \ell_1^{-n+1} \det P(E_1(\ell(\boldsymbol{p})\boldsymbol{p}) - w(\boldsymbol{p})).$$

We have

$$|k(\boldsymbol{p})| \le \ell_1^{-n+1} |\det P|(|F_1(\ell(\boldsymbol{p})\boldsymbol{p})| + |f(\ell(\boldsymbol{p})\boldsymbol{p}) - w(\boldsymbol{p})|)$$

$$\ll \ell_1^{-n+1} |\det P| \frac{A^m}{\det(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)} V$$

(by (2.22), (1.8))

$$\ll \ell_1^{-n+1} \ell_1 \ldots \ell_m A^m V$$

(by (2.21))

$$\ll U^{m-n+1} A^m V$$

by (1.7). Taking (1.9) into account, with $C(h)$ suitably chosen, we have $|k(\boldsymbol{p})| < 1$, and indeed $k(\boldsymbol{p}) = 0$. We may now complete the proof by the argument in Case 2 (a) of the preceding proof. The points $(\ell(\boldsymbol{p})\boldsymbol{p}, w(\boldsymbol{p}))$ lie in an m-dimensional subspace of $\mathbb{R}^h$. In the role of the determinant in Case 2(a), we use

$$\det \begin{bmatrix} \ell(\boldsymbol{p})\boldsymbol{p} & w(\boldsymbol{p}) \\ \ell(\boldsymbol{p}_1)\boldsymbol{p}_1 & w(\boldsymbol{p}_1) \\ \vdots & \\ \ell(\boldsymbol{p}_m)\boldsymbol{p}_m & w(\boldsymbol{p}_m) \end{bmatrix}.$$

3 A lemma with four alternatives.

In the present section we prove a lemma with four alternatives as a stage in the proof of Theorem 1. I have arranged the proof in this way for comparison with the 'three alternatives lemma' (Lemma 17B of [7]). The corresponding result in [3] (formulated a little differently) is Lemma 7.7.

Lemma 5 *Let $h \ge 3, \epsilon > 0$. Let $N \ge C(h, \epsilon)$. Let Δ satisfy*

(3.1) $$1 \le \Delta^{h+1-(1/2h)+\epsilon} \le N.$$

Let $\Lambda = \Delta^{1/h}\mathbb{Z}^h, \Pi = \Delta^{-1/h}\mathbb{Z}^h$, *and let* $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^h$. *Then either*

(i) for every t, *the set* $K_0 + \Lambda + t$ *contains a point* $n^2\boldsymbol{a}_2 + n\boldsymbol{a}_1$ *with* $1 \le n \le N$; *or*

(ii) there is a primitive point $\boldsymbol{p}$ *in* Π *and a natural number* q *with*

$$|\boldsymbol{p}| < N^\delta, q < N^\delta|\boldsymbol{p}|^{-2}, \|q\boldsymbol{a}_i\boldsymbol{p}\| < N^{\delta-i}|\boldsymbol{p}|^{-1} \quad (i = 1, 2); \tag{3.2}$$

or

(iii) there is a pair of linearly independent points $\boldsymbol{p}_1, \boldsymbol{p}_2$ *of* Π, *a natural number* q, *and there are numbers* $a, B, 0 < a < N^\delta, 1 < B < N$, *such that*

$$|\boldsymbol{p}_1|\,|\boldsymbol{p}_2| \ll a^2N^{\delta-1}B, \tag{3.3}$$

$$q \ll a^{-2}B^{-2}N^{2+\delta}, \tag{3.4}$$

$$|\boldsymbol{p}_j|\,\|q\boldsymbol{p}_k\boldsymbol{a}_i\| \ll a^{-1}B^{-1}N^{1-i+\delta} \quad (i = 1, 2;\ (j, k) = (1, 2), (2, 1)); \tag{3.5}$$

or

(iv) there are three linearly independent points $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ *in* Π *with* $|\boldsymbol{p}_j| < N^\delta$ $(j = 1, 2, 3)$ *and a natural number* q *with*

$$q < N^\delta\Delta^2, \quad \|q\boldsymbol{p}_j\boldsymbol{a}_i\| < N^{\delta-i}\Delta^2 \quad (i = 1, 2; \quad j = 1, 2, 3). \tag{3.6}$$

For the proof of Lemma 5, we require the following variant of Lemma 5 of [6].

Lemma 6 *Let* W *be a subspace of* $\mathbb{R}^h, \dim W = 2$, *such that* $\Gamma = W \cap \mathbb{Z}^h$ *is a two-dimensional lattice. Let* $\mathcal{A}$ *be a set of primitive points* $\boldsymbol{p}$ *of* $\Gamma, |\mathcal{A}| \ge 8$. *Suppose that*

$$A/2 < |\boldsymbol{p}| \le A \qquad (\boldsymbol{p} \in \mathcal{A}) \tag{3.7}$$

and $\boldsymbol{e}_1, \boldsymbol{e}_2$ *in* $\mathbb{R}^h$ *and* V_1, V_2 *are such that*

$$9A^2V_j < 1 \qquad (j = 1, 2), \tag{3.8}$$

$$|\boldsymbol{p}\boldsymbol{e}_j - v_j(\boldsymbol{p})| < V_j \qquad (j = 1, 2,\ \boldsymbol{p} \in \mathcal{A}), \tag{3.9}$$

where $v_j(\boldsymbol{p}) \in \mathbb{Z}$. *Then there are linearly independent points* $\boldsymbol{p}_1, \boldsymbol{p}_2$ *of* Γ *for which*

$$|\boldsymbol{p}_1|\,|\boldsymbol{p}_2| \ll A^2|\mathcal{A}|^{-1}, \tag{3.10}$$

$$\max(|\boldsymbol{p}_1|\,\|\boldsymbol{p}_2\boldsymbol{e}_j\|,\ |\boldsymbol{p}_2|\,\|\boldsymbol{p}_1\boldsymbol{e}_j\|) \ll V_jA|\mathcal{A}|^{-1} \quad (j = 1, 2). \tag{3.11}$$

Proof. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be an orthonormal basis of W. We write each $\boldsymbol{p}$ in $\mathcal{A}$ as

$$\boldsymbol{p} = (r\cos\alpha)\boldsymbol{w}_1 + (r\sin\alpha)\boldsymbol{w}_2,\ r = r(\boldsymbol{p}) > 0,\ \alpha = \alpha(\boldsymbol{p}) \in [0, 2\pi).$$

Now for some k, $0 \le k \le 3$, there is a subset $\mathcal{A}'$ of $\mathcal{A}$ having

$$|\mathcal{A}'| \ge |\mathcal{A}|/4,$$
$$\alpha(\boldsymbol{p}) \in [k\pi/2, (k+1)\pi/2] \qquad (\boldsymbol{p} \in \mathcal{A}').$$

Let $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2$ be chosen in $\mathcal{A}'$ so that $\alpha(\boldsymbol{q}_1)$ is least, $\alpha(\boldsymbol{q}_2)$ is greatest, and $\alpha(\boldsymbol{r}_2) - \alpha(\boldsymbol{r}_1)$ is positive and as small as possible. Clearly the $\alpha(\boldsymbol{p})$ $(\boldsymbol{p} \in \mathcal{A}')$ are distinct, and

$$\begin{aligned} 0 < \det(\boldsymbol{r}_1, \boldsymbol{r}_2) &\ll A^2(\alpha(\boldsymbol{r}_2) - \alpha(\boldsymbol{r}_1)) \\ &\ll |\mathcal{A}|^{-1} A^2(\alpha(\boldsymbol{q}_2) - \alpha(\boldsymbol{q}_1)) \\ &\ll |\mathcal{A}|^{-1} \det(\boldsymbol{q}_1, \boldsymbol{q}_2). \end{aligned} \tag{3.12}$$

Let C be the index in Γ of the lattice Γ_0 generated by $\boldsymbol{q}_1, \boldsymbol{q}_2$. Then

$$C\Gamma \subset \Gamma_0. \tag{3.13}$$

We introduce the linear functions $E_j : W \to \mathbb{R}$ defined by

$$E_j(\boldsymbol{q}_1) = v_j(\boldsymbol{q}_1), \quad E_j(\boldsymbol{q}_2) = v_j(\boldsymbol{q}_2)$$

for $j = 1, 2$. We observe that

$$CE_j(\boldsymbol{x}) \in \mathbb{Z} \qquad (\boldsymbol{x} \in \Gamma) \tag{3.14}$$

from (3.13).

Let $f_j(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{e}_j$ and $F_j = f_j - E_j$. Then

$$|F_j(\boldsymbol{q}_i)| < V_j \qquad (1 \le i, j \le 2)$$

from (3.9). Moreover, given $\boldsymbol{p} \in \mathcal{A}', \boldsymbol{p} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$, we have

$$|x_1| = \frac{\det(\boldsymbol{p}, \boldsymbol{q}_2)}{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)} \le 4, \quad |x_2| = \frac{\det(\boldsymbol{p}, \boldsymbol{q}_1)}{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)} \le 4$$

by (3.7) and the choice of $\boldsymbol{q}_1, \boldsymbol{q}_2$. Hence

$$|F_j(\boldsymbol{p})| < 8V_j. \tag{3.15}$$

The integer

$$k_j(\boldsymbol{p}) = C(E_j(\boldsymbol{p}) - v_j(\boldsymbol{p}))$$

satisfies

$$|k_j(\boldsymbol{p})| \le C(|f_j(\boldsymbol{p}) - v_j(\boldsymbol{p})| + |F_j(\boldsymbol{p})|) < 9CV_j \qquad (\boldsymbol{p} \in \mathcal{A}')$$

from (3.9), (3.15).

Taking $\boldsymbol{s}_1, \boldsymbol{s}_2$ to be a basis of Γ, we see that

$$C = \frac{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)}{\det(\boldsymbol{s}_1, \boldsymbol{s}_2)} \le \det(\boldsymbol{q}_1, \boldsymbol{q}_2) \le A^2,$$

and in view of (3.8),

$$|k_j(\boldsymbol{p})| < 9A^2 V_j < 1.$$

Hence $k_j(\boldsymbol{p}) = 0$. In particular,

$$E_j(\boldsymbol{p}) \in \mathbb{Z} \qquad (j = 1, 2) \tag{3.16}$$

for all $\boldsymbol{p}$ in $\mathcal{A}'$.

The set Γ_1 of $\boldsymbol{p}$ in Γ satisfying (3.16) is clearly a two-dimensional lattice, and indeed

$$\det \Gamma_1 \le \det(\boldsymbol{r}_1, \boldsymbol{r}_2).$$

By Minkowski's theorem, there are linearly independent points $\boldsymbol{p}_1, \boldsymbol{p}_2$ in Γ_1 with

$$\begin{aligned} |\boldsymbol{p}_1|\,|\boldsymbol{p}_2| &\ll \det \Gamma_1 \le \det(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &\ll |\mathcal{A}|^{-1} \det(\boldsymbol{q}_1, \boldsymbol{q}_2) \ll |\mathcal{A}|^{-1} A^2, \end{aligned} \tag{3.17}$$

on taking into account (3.12), (3.7).

Now let $u_{j,i} = E_j(\boldsymbol{p}_i)$. Then $u_{j,i}$ is an integer, and

$$\begin{aligned} |\boldsymbol{p}_1|\,|\boldsymbol{p}_2 \boldsymbol{e}_j - u_{j,2}| &= |\boldsymbol{p}_1|\,|F_j(\boldsymbol{p}_2)| \\ &\le |\boldsymbol{p}_1| \left(\frac{\det(\boldsymbol{p}_2, \boldsymbol{q}_2)}{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)} |F_j(\boldsymbol{q}_1)| + \frac{\det(\boldsymbol{p}_2, \boldsymbol{q}_1)}{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)} |F_j(\boldsymbol{q}_2)| \right) \end{aligned}$$

(by the argument leading to (3.15))

$$\begin{aligned} &\le \frac{|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|(|\boldsymbol{q}_2| + |\boldsymbol{q}_1|)V_j}{\det(\boldsymbol{q}_1, \boldsymbol{q}_2)} \\ &\ll |\mathcal{A}|^{-1} A V_j \end{aligned}$$

in view of (3.17). The same bound holds with $\boldsymbol{p}_1, \boldsymbol{p}_2$ interchanged. This completes the proof of Lemma 6.

Proof of Lemma 5. Suppose that alternative (i) does not hold. By a slight variant of the proof of [3], Lemma 7.5, there are numbers a and B such that

$$\Delta^{-1} \ll a \ll N^{\delta}, \tag{3.18}$$

$$B \gg N^{1-\delta}\Delta^{-1}a^{-1}, \tag{3.19}$$

and there is a set $\mathcal{B}$ of primitive points of Π with

$$a < |\boldsymbol{p}| \leq 2a \qquad (\boldsymbol{p} \in \mathcal{B}), \tag{3.20}$$

$$|\mathcal{B}| \gg NB^{-1}(\log N)^{-2}. \tag{3.21}$$

Further, for each $\boldsymbol{p}$ in $\mathcal{B}$ there are integers $q = q(\boldsymbol{p}), v_1 = v_1(\boldsymbol{p}), v_2 = v_2(\boldsymbol{p})$ satisfying

$$1 \leq q < a^{-2}B^{-2}N^{2+\delta}, \tag{3.22}$$

$$(q, v_1, v_2) = 1, (q, v_2) < N^{\delta}a^{-1}, \tag{3.23}$$

$$|q\boldsymbol{a}_i\boldsymbol{p} - v_i| < a^{-1}B^{-2}N^{2-i+\delta} \quad (i = 1, 2). \tag{3.24}$$

Let us write $s = s(\boldsymbol{p}) = (q, v_2), r = r(\boldsymbol{p}) = qs^{-1}, v = v(\boldsymbol{p}) = v_2s^{-1}$. Then we note that

$$r \geq 1, s \geq 1, rs < a^{-2}B^{-2}N^{2+\delta}, \tag{3.25}$$

$$s|r\boldsymbol{a}_2\boldsymbol{p} - v| < a^{-1}B^{-2}N^{\delta}, (r, v) = 1, \tag{3.26}$$

$$|sr\boldsymbol{a}_1\boldsymbol{p} - v_1| < a^{-1}B^{-2}N^{1+\delta}, (s, v_1) = 1, \tag{3.27}$$

$$s < N^{\delta}a^{-1}. \tag{3.28}$$

There are now two cases to consider. Suppose first that

$$B \geq N^{1-3\epsilon^2}. \tag{3.29}$$

Take any $\boldsymbol{p} \in \mathcal{B}$. Then alternative (ii) holds with this choice of $\boldsymbol{p}$ and $q = q(\boldsymbol{p})$. For

$$q < a^{-2}B^{-2}N^{2+\delta} < a^{-2}N^{\delta} < |\boldsymbol{p}|^{-2}N^{\delta}$$

by (3.22), (3.29), (3.20), while

$$\begin{aligned}\|q\boldsymbol{a}_i\boldsymbol{p}\| < a^{-1}B^{-2}N^{2-i+\delta} &< a^{-1}N^{-i+\delta}\\ &< |\boldsymbol{p}|^{-1}N^{-i+\delta} \quad (i=1,2)\end{aligned}$$

by (3.24), (3.29), (3.20).

Now suppose that (3.29) is false. Clearly (3.21) yields a subset $\mathcal{B}'$ of $\mathcal{B}$ with

$$|\mathcal{B}'| \geq |\mathcal{B}|\, N^{-\epsilon^2} \geq N^{2\epsilon^2},\ \ U/2 < r(\boldsymbol{p}) \leq U < a^{-2}B^{-2}N^{2+\delta} \quad (\boldsymbol{p}\in\mathcal{B}').$$

We apply Theorem 2 with ϵ^2 in place of ϵ,

$$\mathcal{A} = \Delta^{1/h}\mathcal{B}',\ \boldsymbol{e} = \boldsymbol{a}_2\Delta^{-1/h},\ \ell(\boldsymbol{p}) = r(\boldsymbol{p}),\ w(\boldsymbol{p}) = v(\boldsymbol{p}).$$

Thus we may take

$$\begin{aligned}A = 2\Delta^{1/h}a, \quad & U < a^{-2}B^{-2}N^{2+\delta},\\ & V = a^{-1}B^{-2}N^{\delta}, M = UA,\end{aligned}$$

in view of (3.20), (3.25), (3.26). We must verify (1.2), (1.3). We have

$$\begin{aligned}M^{h-1+\epsilon^2}AV &\ll U^{h-1}A^hVN^{\delta}\\ &\ll a^{-h+1}B^{-2h}N^{2h-2+\delta}\Delta\\ &\ll \Delta^{2h+1}N^{-2+\delta} \ll N^{-\delta}\end{aligned} \tag{3.30}$$

from (3.19), (3.18), (3.1). Moreover,

$$\begin{aligned}|\mathcal{A}|M^{-2\epsilon^2}(M^hV)^{-h/(h+1)} &\\ &\gg N^{1-\delta}B^{-1}(a^{-h-1}B^{-2h-2}N^{2h}\Delta)^{-h/(h+1)}\\ &\gg N^{1-2h^2/(h+1)-\delta}B^{2h-1}a^h\Delta^{-h/(h+1)}\\ &\gg N^{2h-2h^2/(h+1)-\delta}\Delta^{-2h+1-h/(h+1)} \gg M^{\delta}\end{aligned}$$

from (3.21), (3.19), (3.18), (3.1). This establishes that (1.2), (1.3) hold. Thus there is a subset $\mathcal{A}_1$ of $\mathcal{A}$ with

$$|\mathcal{A}_1| \gg M^{2\epsilon^2}$$

and $r(\boldsymbol{p}) = r$ for all $\boldsymbol{p}$ in $\mathcal{A}_1$.

We now use Theorem 3 to find a subset $\mathcal{A}_2$ of $\mathcal{A}$ with

$$|\mathcal{A}_2| \gg |\mathcal{A}| M^{-\delta} \gg N^{1-\delta} B^{-1}$$

and $r(\boldsymbol{p}) = r$ for all $\boldsymbol{p}$ in $\mathcal{A}_2$. We take $\mathcal{A}, \boldsymbol{e}, \ell(\boldsymbol{p}), w(\boldsymbol{p}), A, U, V$ and M as above. We have $2 \le m \le h$. Since $\mathcal{A}_1$ consists of primitive points, we can certainly take $n \ge 2$. It follows that

$$U^{1+m-n} A^m V \ll M^{h-1} A V \ll N^{-\delta}.$$

Having 'fixed r' on the set $\mathcal{B}_1 = \Delta^{-1/h} \mathcal{A}_2$ in (3.25)–(3.28), we now 'fix s'. In view of (3.20), (3.27), (3.28) we may apply Lemma 4 with $\mathcal{B}_1$ in place of $\mathcal{A}$, $\boldsymbol{e} = r\boldsymbol{a}_1$, $\ell(\boldsymbol{p}) = s(\boldsymbol{p})$, $w(\boldsymbol{p}) = v_1(\boldsymbol{p})$, and with

$$Z = (2a)^t, \;\; U = N^\delta a^{-1}, \;\; V = a^{-1} B^{-2} N^{1+\delta},$$

where t is the dimension of $\operatorname{Span} \mathcal{B}_1$. Now

$$\begin{aligned} Z U^t V \Delta N^\delta &\ll (2a)^t (N^\delta a^{-1})^t a^{-1} B^{-2} N^{1+\delta} \Delta \\ &\ll \Delta^3 N^{-1+\delta} \ll N^{-\delta} \end{aligned}$$

from (3.19), (3.18), (3.1). Thus there is a subset $\mathcal{B}_2$ of $\mathcal{B}_1$ with

$$|\mathcal{B}_2| \gg |\mathcal{B}_1| N^{-\delta} \gg N^{1-\delta} B^{-1}, \tag{3.31}$$

with $s(\boldsymbol{p})$, and indeed $q(\boldsymbol{p})$, constant throughout $\mathcal{B}_2$:

$$q(\boldsymbol{p}) = q.$$

If $\mathcal{B}_2$ contains three linearly independent points, it is clear that alternative (iv) of Lemma 5 holds. It remains to consider the case where $W = \operatorname{Span} \mathcal{B}_2$ has dimension 2. In that case, we apply Lemma 6 with ϵ^2 in place of ϵ, $\Delta^{1/h} \mathcal{B}_2$ in place of $\mathcal{A}$, taking $\boldsymbol{e}_j = \Delta^{-1/h} q \boldsymbol{a}_j$ $(j = 1, 2)$, so that (3.7)–(3.9) hold with

$$A = 2\Delta^{1/h} a, \;\; V_j = a^{-2} B^{-2} N^{2-j+\delta}.$$

The condition (3.8) is satisfied, since

$$\Delta^{2/h} a^2 V_j \ll \Delta^{2/h+2} N^{-1+\delta} \ll N^{-\delta} \quad (j = 1, 2)$$

from (3.19), (3.18), (3.1). Let $\boldsymbol{p}_1', \boldsymbol{p}_2'$ be the independent points of $W \cap \mathbb{Z}^h$ provided by Lemma 6, and $\boldsymbol{p}_i = \Delta^{-1/h} \boldsymbol{p}_i'$. Then (3.3), (3.4), (3.5) follow from (3.10), (3.31), (3.22), (3.11). Thus alternative (iii) holds, and the proof of Lemma 5 is complete.

4 Proof of Theorem 1.

Lemma 7 *Let $h \geq 1$, $\epsilon > 0$, $N > C(h, \epsilon)$. Let Λ be an h-dimensional lattice in $\mathbb{R}^h$ with*

$$K_0 \cap \Lambda = \{\mathbf{0}\},$$

$$d(\Lambda)^{h+1+\epsilon} \leq N. \tag{4.1}$$

For any $\boldsymbol{a}_1, \boldsymbol{a}_2$ in $\mathbb{R}^h$, there is a natural number $n \leq N$ such that

$$n^2 \boldsymbol{a}_2 + n\boldsymbol{a}_1 \in K_0 + \Lambda.$$

Proof. This is Theorem 7.2 of [3]. It contains the admissibility of $1/(h^2+h)$ as a special case, as we see on taking $\Lambda = N^{1/(h^2+h)-\epsilon}\mathbb{Z}^h$. (The methods of the present paper do not seem to be strong enough to sharpen Lemma 7 for a general lattice.)

The following lemma is a refinement of [3], Lemma 7.9. We give the proof in detail for the convenience of readers. The orthogonal complement of a subspace T in $\mathbb{R}^h$ is denoted by $T^\perp$.

Lemma 8 *Let Λ be an h-dimensional lattice in $\mathbb{R}^h$ with polar latice Π. Let Π' be a t-dimensional lattice contained in Π, let $T = \mathrm{Span}\,\Pi'$, and let $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_t$ be a linearly independent set in Π'. Then there is a natural number c,*

$$c \ll \det(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_t)/d(\Pi'), \tag{4.2}$$

having the following property. Given $\boldsymbol{a}$ in $\mathbb{R}^h, c\boldsymbol{a}$ may be written in the form

$$c\boldsymbol{a} = \boldsymbol{\ell} + \boldsymbol{s} + \boldsymbol{b}, \tag{4.3}$$

where $\boldsymbol{\ell} \in \Lambda, \boldsymbol{s} \in T^\perp$ and

$$|\boldsymbol{b}| \ll d(\Pi')^{-1} \max_{1 \leq i \leq t} |\boldsymbol{p}_1| \ldots |\boldsymbol{p}_{i-1}| \, \|\boldsymbol{p}_i \boldsymbol{a}\| \, |\boldsymbol{p}_{i+1}| \ldots |\boldsymbol{p}_t|. \tag{4.4}$$

Proof. Let $\lambda_1, \ldots, \lambda_t$ be the successive minima of Π' with respect to K_0 and let $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_t$ be linearly independent points of Π' with $|\boldsymbol{q}_j| = \lambda_j$. By Minkowski's theorem,

$$1 \leq v := \frac{\det(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_t)}{d(\Pi')} \leq \frac{|\boldsymbol{q}_1| \ldots |\boldsymbol{q}_t|}{d(\Pi')} \ll 1. \tag{4.5}$$

Arguing as in the proof of Lemma 7.8 of [3], we find points $\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_t$ of $v^{-1}\Lambda$ such that

$$\boldsymbol{\ell}_i \boldsymbol{q}_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases} \tag{4.6}$$

Let $\boldsymbol{w}_1, \dots, \boldsymbol{w}_t$ be an orthonormal basis of T, and write

$$\boldsymbol{p}_j = p_{j1}\boldsymbol{w}_1 + \cdots + p_{jt}\boldsymbol{w}_t, \quad \boldsymbol{q}_j = q_{j1}\boldsymbol{w}_1 + \cdots + q_{jt}\boldsymbol{w}_t.$$

There are integers c_{ij} such that

$$v\boldsymbol{p}_j = c_{j1}\boldsymbol{q}_1 + \cdots + c_{jt}\boldsymbol{q}_t \qquad (j = 1, \dots, t).$$

Write $C = [c_{ij}], c = |\det C|$, and let C_{ij} be the cofactor of c_{ij} in C. Obviously

$$v^t \det(\boldsymbol{p}_1, \dots, \boldsymbol{p}_t) = c \det(\boldsymbol{q}_1, \dots, \boldsymbol{q}_t).$$

Taking (4.5) into account, we obtain (4.2).

We now fix j and solve the t equations

$$c_{j1}q_{1i} + \cdots + c_{jt}q_{ti} = vp_{ji} \qquad (i = 1, \dots, t)$$

for c_{js} by Cramer's rule. This yields

$$\begin{aligned} c_{js} &\ll \frac{|\boldsymbol{q}_1| \dots |\boldsymbol{q}_{s-1}|\, |\boldsymbol{p}_j|\, |\boldsymbol{q}_{s+1}| \dots |\boldsymbol{q}_t|}{\det(\boldsymbol{q}_1, \dots, \boldsymbol{q}_t)} \\ &\ll |\boldsymbol{p}_j|/|\boldsymbol{q}_s| \end{aligned}$$

by (4.5). It follows that for $1 \leq i,\, r \leq t$,

$$\begin{aligned} C_{ir} &\ll \frac{|\boldsymbol{p}_1| \dots |\boldsymbol{p}_{i-1}|\, |\boldsymbol{p}_{i+1}| \dots |\boldsymbol{p}_t|}{|\boldsymbol{q}_1| \dots |\boldsymbol{q}_{r-1}|\, |\boldsymbol{q}_{r+1}| \dots |\boldsymbol{q}_t|} \\ &\ll \frac{|\boldsymbol{p}_1| \dots |\boldsymbol{p}_{i-1}|\, |\boldsymbol{p}_{i+1}| \dots |\boldsymbol{p}_t|\, |\boldsymbol{q}_r|}{d(\Pi')}. \end{aligned} \tag{4.7}$$

We are now ready to deduce the representation (4.3), (4.4). We have

$$v\boldsymbol{p}_j\boldsymbol{a} = vx_j + P_j,$$

where $x_j \in \mathbb{Z}$ and $P_j \ll \|\boldsymbol{p}_j \boldsymbol{a}\|$. That is,

$$c_{j1}\boldsymbol{q}_1\boldsymbol{a} + \cdots + c_{jt}\boldsymbol{q}_t\boldsymbol{a} = vx_j + P_j \quad (j = 1, \ldots, t).$$

For a fixed i, we multiply the j-th equation by C_{ji} and add to get

$$c\boldsymbol{q}_i\boldsymbol{a} = vy_i + V_i,$$

where $y_i \in \mathbb{Z}$ and

$$\begin{aligned} V_i &\ll \max_j |C_{ji}P_j| \\ &\ll \frac{|\boldsymbol{q}_i|}{d(\Pi')} \max_j |\boldsymbol{p}_1| \ldots |\boldsymbol{p}_{j-1}| \, \|\boldsymbol{p}_j\boldsymbol{a}\| \, |\boldsymbol{p}_{j+1}| \ldots |\boldsymbol{p}_t| \end{aligned} \tag{4.8}$$

in view of (4.7).

Define $\boldsymbol{\ell} = v(y_1\boldsymbol{\ell}_1 + \cdots + y_t\boldsymbol{\ell}_t)$; then $\boldsymbol{\ell} \in \Lambda$ and

$$\boldsymbol{q}_i(c\boldsymbol{a} - \boldsymbol{\ell}) = vy_i + V_i - vy_i = V_i \quad (i = 1, \ldots, t).$$

We now decompose $c\boldsymbol{a} - \boldsymbol{\ell}$ into

$$c\boldsymbol{a} - \boldsymbol{\ell} = \boldsymbol{b} + \boldsymbol{s} \qquad (\boldsymbol{b} \in T, \boldsymbol{s} \in T^{\perp})$$

and give a bound for $|\boldsymbol{b}|$. We have

$$\boldsymbol{q}_i\boldsymbol{b} = \boldsymbol{q}_i(\boldsymbol{b} + \boldsymbol{s}) = V_i \quad (i = 1, \ldots, t)$$

because $\boldsymbol{q}_i \in T$. Writing

$$\boldsymbol{b} = b_1\boldsymbol{w}_1 + \cdots + b_t\boldsymbol{w}_t,$$

we have the equations

$$q_{i1}b_1 + \cdots + q_{it}b_t = V_i \quad (i = 1, \ldots, t)$$

for $b_1, \ldots, b_t$. Solving by Cramer's rule,

$$\det(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_t)b_j = \pm(Q_{1j}V_1 + \cdots + Q_{tj}V_t), \tag{4.9}$$

where Q_{ij} is the cofactor of q_{ij} in $[q_{rs}]$. Now

$$|Q_{ij}| \ll \prod_{\ell \neq i} |\boldsymbol{q}_\ell|. \tag{4.10}$$

We obtain

$$|b_j| \ll d(\Pi')^{-1} \sum_{i=1}^{t} \left(\prod_{\ell \neq i} |\boldsymbol{q}_\ell| \right) |\boldsymbol{q}_i| d(\Pi')^{-1} \max_k |\boldsymbol{p}_i| \dots |\boldsymbol{p}_{k-1}| \, \|\boldsymbol{p}_k \boldsymbol{a}\| \, |\boldsymbol{p}_{k+1}| \dots |\boldsymbol{p}_t|$$

on combining (4.8)–(4.10) and recalling (4.5). Now the lemma follows on a further application of (4.5).

Proof of Theorem 1. Let $\epsilon > 0$, $h \geq 3$, $N > C(h, \epsilon)$. Take $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_h)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_h) \in \mathbb{R}^h$. Suppose that there is no natural number $n \leq N$ such that

$$\|\alpha_i n^2 + \beta_i n\| < N^{\epsilon - \varphi} \quad (i = 1, \dots, h), \tag{4.11}$$

where $\varphi^{-1} = h^2 + h - 1/2$. Write $\boldsymbol{a}_2 = N^{\varphi - \epsilon} \boldsymbol{\alpha}$, $\boldsymbol{a}_1 = N^{\varphi - \epsilon} \boldsymbol{\beta}$, $\Lambda = N^{\varphi - \epsilon} \mathbb{Z}^h$. Then there is no natural number $n \leq N$ such that

$$n^2 \boldsymbol{a}_2 + n \boldsymbol{a}_1 \in K_0 + \Lambda.$$

Moreover, Λ satisfies the hypotheses of Lemma 5 with $\Delta = N^{h(\varphi - \epsilon)}$. Hence one of the cases (ii), (iii) or (iv) must hold. We apply Lemma 8, taking Π' to be the lattice generated by $\boldsymbol{p}$ in Case (ii); by $\boldsymbol{p}_1, \boldsymbol{p}_2$ in Case (iii); and by $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ in Case (iv). Let $\Lambda' = \Lambda \cap T^\perp$. In each case, we have the inequality

$$d(\Lambda') \ll d(\Pi') \Delta$$

whenever $\dim T < h$ ([3], Lemma 7.8). Our choices of $\boldsymbol{a}$ are $\boldsymbol{a}_i = q^i \boldsymbol{a}_i$ for $i = 1, 2$. We obtain the representation

$$c q^i \boldsymbol{a}_i = \boldsymbol{\ell}_i + \boldsymbol{s}_i + \boldsymbol{b}_i \quad (i = 1, 2),$$

where $\boldsymbol{\ell}_i \in \Lambda, \boldsymbol{s}_i \in T^\perp$ and

$$c \ll 1, \quad |\boldsymbol{b}_i| \ll |\boldsymbol{p}|^{-1} \|\boldsymbol{p} q^i \boldsymbol{a}_i\| \tag{4.12}$$

in Case (ii),

$$c \ll |\boldsymbol{p}_1| \, |\boldsymbol{p}_2| / d(\Pi'), \ |\boldsymbol{b}_i| \ll d(\Pi')^{-1} |\boldsymbol{p}_1| \, \|\boldsymbol{p}_2 q^i \boldsymbol{a}\| \tag{4.13}$$

in Case (iii),

$$c \ll |\boldsymbol{p}_1| \, |\boldsymbol{p}_2| \, |\boldsymbol{p}_3| / d(\Pi'), \ |\boldsymbol{b}_i| \ll d(\Pi')^{-1} |\boldsymbol{p}_1| \, |\boldsymbol{p}_2| \, \|\boldsymbol{p}_3 q^i \boldsymbol{a}_i\| \tag{4.14}$$

in Case (iv). (We permit renumbering of the $\boldsymbol{p}_i$ in Cases (iii), (iv).)

We now apply Lemma 8 in the space $T^\perp$, whose dimension we denote by t. We replace ϵ by ϵ^2, Λ by $2\Lambda'$, $\boldsymbol{a}_i$ by $2c^{i-1}\boldsymbol{s}_i$ and N by $d(2\Lambda')^{t+1}N^\delta$. Thus if $t > 0$ there is a natural number x,

$$x \le d(2\Lambda')^{t+1}N^\delta \ll d(\Pi')^{t+1}\Delta^{t+1}N^\delta, \tag{4.15}$$

such that

$$2x^2c\boldsymbol{s}_2 + 2x\boldsymbol{s}_1 \in 2\Lambda' + K_0.$$

This implies

$$x^2c\boldsymbol{s}_2 + x\boldsymbol{s}_1 \in \Lambda + \frac{1}{2}K_0. \tag{4.16}$$

If $t = 0$, we take $x = 1$. Of course (4.16) holds, since $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \mathbf{0}$.

Now let $n = xcq$. We shall show that

$$n \ll N^{1-\delta}, \tag{4.17}$$

$$x^ic^{i-1}|\boldsymbol{b}_i| \ll N^{-\delta} \quad (i = 1, 2). \tag{4.18}$$

Suppose for a moment that (4.17), (4.18) hold. We see that the natural number $n \le N$ satisfies

$$\begin{aligned} n^2\boldsymbol{a}_2 + n\boldsymbol{a}_1 &= x^2c(\boldsymbol{\ell}_2 + \boldsymbol{s}_2 + \boldsymbol{b}_2) + x(\boldsymbol{\ell}_1 + \boldsymbol{s}_1 + \boldsymbol{b}_1) \\ &= (x^2c\boldsymbol{s}_2 + x\boldsymbol{s}_1) + (x^2c\boldsymbol{b}_2 + x\boldsymbol{b}_1) + \boldsymbol{\ell}, \end{aligned}$$

where $\boldsymbol{\ell} \in \Lambda$. Taking (4.16)–(4.18) into account,

$$n^2\boldsymbol{a}_2 + n\boldsymbol{a}_1 \in \Lambda + K_0.$$

This contradicts our hypothesis. Hence there must be a solution of (4.11) after all, and the proof is complete.

It remains to prove (4.17), (4.18). Consider Case (ii) first. Here $t = h-1$,

$$\begin{aligned} n = xcq &\ll d(\Pi')^h\Delta^h qN^\delta \\ &\ll |\boldsymbol{p}|^h\Delta^h|\boldsymbol{p}|^{-2}N^\delta \ll \Delta^hN^\delta \ll N^{1-\delta} \end{aligned}$$

from (4.15), (4.12), (3.2), (3.1). Further

$$\begin{aligned} x^ic^{i-1}|\boldsymbol{b}_i| &\ll d(\Pi')^{hi}\Delta^{hi}|\boldsymbol{p}|^{-1}q^{i-1}\|q\boldsymbol{p}\boldsymbol{a}_i\| \\ &\ll |\boldsymbol{p}|^{hi-1}\Delta^{hi}|\boldsymbol{p}|^{-2i+1}N^{\delta-i} \\ &\ll (\Delta^hN^{-1+\delta})^i \ll N^{-\delta}, \end{aligned}$$

again from (4.15), (4.12), (3.2), (3.1).

Now consider Case (iii). Here $t = h - 2$,

$$n = xcq \ll d(\Pi')^{h-1}\Delta^{h-1}|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|d(\Pi')^{-1}a^{-2}B^{-2}N^{2+\delta}$$
$$\ll (|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|)^{h-1}\Delta^{h-1}a^{-2}B^{-2}N^{2+\delta}$$

from (4.15), (4.13), (3.4), and since

$$d(\Pi') \leq |\boldsymbol{p}_1|\,|\boldsymbol{p}_2|. \tag{4.19}$$

Recalling (3.3),

$$n \ll (a^2N^{-1}B)^{h-1}\Delta^{h-1}a^{-2}B^{-2}N^{2+\delta}$$
$$\ll a^{2h-4}B^{h-3}N^{-h+3+\delta}\Delta^{h-1}$$
$$\ll \Delta^{h-1}N^{\delta} \ll N^{1-\delta}$$

since $a < N^{\delta}, B < N$. Similarly,

$$x^i c^{i-1}|\boldsymbol{b}_i| \ll d(\Pi')^{(h-1)i}\Delta^{(h-1)i}N^{\delta}(|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|)^{i-1}d(\Pi')^{-i}|\boldsymbol{p}_1|q^{i-1}\|q\boldsymbol{p}_2\boldsymbol{a}_i\|$$

(from (4.15), (4.13))

$$\ll (|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|)^{(h-1)i-1}\Delta^{(h-1)i}(a^{-2}B^{-2}N^2)^{i-1}B^{-1}N^{1-i+\delta}$$

(from (4.19), (3.4), (3.5))

$$\ll \Delta^{(h-1)i}(a^2N^{-1}B)^{(h-1)i-1}(a^{-2}B^{-2}N^2)^{i-1}a^{-1}B^{-1}N^{1-i+\delta}$$
$$\ll (\Delta^{h-1}a^{2h-5}B^{h-3}N^{-h+2+\delta})^i \ll (\Delta^{h-1}N^{-1+\delta})^i$$
$$\ll N^{-\delta}$$

from (3.3), (3.1).

Finally, consider Case (iv). Here $t = h - 3$. Suppose first that $t > 0$. Then

$$n = xcq \ll d(\Pi')^{h-2}\Delta^{h-2}|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|\,|\boldsymbol{p}_3|d(\Pi')^{-1}N^{\delta}\Delta^2$$
$$\ll \Delta^h N^{\delta} \ll N^{1-\delta}$$

from (4.15), (4.14), (3.6) and the bounds

$$d(\Pi') \leq |\boldsymbol{p}_1|\,|\boldsymbol{p}_2|\,|\boldsymbol{p}_3| < N^{\delta}. \tag{4.20}$$

Similarly,

$$x^i c^{i-1}|\boldsymbol{b}_i| \ll d(\Pi')^{(h-2)i}\Delta^{(h-2)i}N^{\delta}(|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|\,|\boldsymbol{p}_3|)^{i-1}d(\Pi')^{-i}|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|q^{i-1}\|q\boldsymbol{p}_3\boldsymbol{a}_i\|$$

(from (4.15), (4.14))

$$\ll \Delta^{(h-2)i+2(i-1)+2}N^{\delta-i}$$

(from (4.20), (3.6))

$$\ll (\Delta^h N^{-1+\delta})^i \ll N^{-\delta}.$$

We argue a little differently in Case (iv) if $h = 3, t = 0$. We have $\Pi' = \Pi$,

$$\begin{aligned} n = cq &\ll |\boldsymbol{p}_1|\,|\boldsymbol{p}_2|\,|\boldsymbol{p}_3|d(\Pi')^{-1}\Delta^2 N^{\delta} \\ &\ll \Delta^3 N^{\delta} \ll N^{1-\delta} \end{aligned}$$

from (4.14), (3.6), (4.20). Similarly,

$$\begin{aligned} c^{i-1}|\boldsymbol{b}_i| &\ll (|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|\,|\boldsymbol{p}_3|)^{i-1}d(\Pi')^{-i}|\boldsymbol{p}_1|\,|\boldsymbol{p}_2|q^{i-1}\|q\boldsymbol{p}_3\boldsymbol{a}_i\| \\ &\ll \Delta^{i+2(i-1)+2}N^{-i+\delta} \ll N^{-\delta} \end{aligned}$$

from (4.14), (3.6), (4.20). We have now obtained (4.17), (4.18) in all cases, and the proof of Theorem 1 is complete.

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Department of Mathematics
Brigham Young University
Provo, Utah 84602
U. S. A.
baker@math.byu.edu