INTEGERS AND LIMITS REVIEW PROBLEMS

MATH 290

(1) Let $a, b, r, q \in \mathbb{Z}$, with $a = bq + r$. Prove that $gcd(a, b) = gcd(b, r)$.

(2) Let $a, b \in \mathbb{N}$. Prove that if $gcd(a, b) > 1$, then either $b|a$ or b is not prime.

(3) Let $a, b \in \mathbb{Z}$, which are not both zero. Prove that if c is a common divisor of a and b, then $c|gcd(a, b)$.

(4) For each of the following pairs of integers a, b, use the Euclidean algorithm to find $gcd(a, b)$, and the extended Euclidean algorithm to find $s, t \in \mathbb{Z}$ such that $gcd(a, b) = sa + tb$:

  (a) $a = 374, b = 946$

  (b) $a = 1125, b = 2744$

  (c) $a = 484, b = 1155$

(5) Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $a|bc$, and $gcd(a, b) = 1$, then $a|c$.

(6) Let $a, b \in \mathbb{Z}$, which are not both zero. Show that there exist an infinite number of integers $s, t \in \mathbb{Z}$ such that $gcd(a, b) = sa + tb$. (You may use the theorem that tells you that there is at least one such pair of integers.)

(7) Let $a, b, c, m, n \in \mathbb{Z}$ with $m, n \geq 2$. Prove that if $a \equiv b \pmod{m}$ and $a \equiv c \pmod{n}$, then $b \equiv c \pmod{gcd(m, n)}$.

(8) Let $a \in \mathbb{Z}$ be odd. Prove that $gcd(a, a + 2) = 1$.

(9) Prove that the sequence $\{e^n + 1\}$ diverges to infinity.

(10) Prove that the sequence $\left\{\frac{n+1}{3n-1}\right\}$ converges to $\frac{1}{3}$.

(11) Prove that the sequence $\{1 + (-2)^n\}$ diverges.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \overline{\mathbb{Q}}. \end{cases}$$

Prove that $\lim_{x\to 0} f(x) = 0$. Does $\lim_{x\to a} f(x)$ exist for any other $a \neq 0$?

(13) Prove that $\lim_{x\to 2} \frac{3}{2}x + 1 = 4$.

(14) Prove that $\lim_{x\to 1} \frac{1}{5x-4} = 1$.