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Name:

Instructor:

## Math 290 Sample Exam 2

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice—mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly.

### **True-false** questions

- 1. For every natural number n, the integer  $2n^2 4n + 31$  is prime.
- 2. For every real number x, it is not the case that  $x^4 < x < x^2$ .
- 3. The number of length 11 lists, from the alphabet  $\{A, B, \dots, Z\}$ , is  $11^{26}$ .
- 4. The number of length 11 lists, from the alphabet  $\{A, B, \ldots, Z\}$ , with no repetitions, is 26!/11!.
- 5. If you expand  $(7a + 3b)^{11}$ , the coefficient of  $ab^{10}$  is  $7 \cdot 3^{10} \cdot 11$ .
- 6. If proving a statement for all integers, by induction, then you don't need a base case.
- 7. To prove the statement "There exists a unique integer n, such that 7 < n < 9" you just need to say "The integer 8 works."
- 8. The number of subsets of cardinality 6, from a 9 element set, is  $\binom{9}{6}$ .
- 9. If A and B are sets, then  $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ .
- 10. When proving  $A \subseteq B$ , you assume  $x \in A$  and  $x \in B$ .

#### Multiple choice section

On all problems, choose the most complete correct answer.

11. If n is the number of length 5 lists with no repetitions that can be made from the symbols A, B, C, D, E, F, then

a)  $0 \le n \le 10$ . b)  $10 < n \le 100$ . c)  $100 < n \le 400$ . d)  $400 < n \le 800$ . e) 800 < n.

12. How many lists of length 4 from the symbols  $\{D, O, N, U, T\}$  (with repetition allowed) contain at least one O?

a)  $0 \le n \le 100$ . b)  $100 < n \le 200$ . c)  $200 < n \le 300$ . d)  $300 < n \le 400$ . e) 400 < n.

13. The value of  $\binom{10}{5}$  is: a) 10. b) 15. c) 72. d) 252. e) 30240. f) 100000. g) 3628800.

14. How many 8 digit binary strings have a 1 in their second and fourth digits. (For instance, 01110101 and 11111111 are two such strings.)

a) 8 b)  $8^6$  c) 8! d) 8!/2!e)  $8^8 - 8^2$  f) 8!/(2!6!) g) None of the above.

- 15. If we want to show that an integer n satisfying some property is unique, then we could:
  - a) Assume n is unique, and derive a contradiction.
  - b) Assume n is not unique, and derive a contradiction.
  - c) Assume m, n are both integers with the property, and derive a contradiction.
  - d) Assume m, n are both integers with the property, and show m = n.
  - e) Three of the above answers.
  - f) Two of the above answers.
  - g) None of the above.

#### 16. The negation of the statement:

For every  $x \in \mathbb{Z}$  there exists a  $y \in \mathbb{Z}$  such that  $y^2 > x$ .

- is
- a) For every  $x \in \mathbb{Z}$ , there does not exist a  $y \in \mathbb{Z}$  such that  $y^2 > x$ .
- b) For every  $x \in \mathbb{Z}$  there does not exist a  $y \in \mathbb{Z}$  such that  $y^2 \leq x$ .
- c) There exists an  $x \in \mathbb{Z}$  such that for all  $y \in \mathbb{Z}, y^2 > x$ .
- d) There exists an  $x \in \mathbb{Z}$  such that for all  $y \in \mathbb{Z}$ ,  $y^2 \leq x$ .
- e) There exists a  $y \in \mathbb{Z}$  such that for every  $x \in \mathbb{Z}, y^2 > x$ .
- f) For all  $y \in \mathbb{Z}$ , there exists  $x \in \mathbb{Z}$  such that  $y^2 \leq x$ .
- g) None of the above.

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17. Which of the following sets equals  $\{2x + 3y : x, y \in \mathbb{N}\}$ .

a) 
$$\mathbb{Z}$$
 b)  $\mathbb{N}$  c)  $2\mathbb{Z}$  d)  $\{2, 3, 4, 5, ...\}$  e)  $\{5, 7, 8, 9, ...\}$  e) None of the above.

18. Evaluate the following proof that  $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ .

**Proof.** Take  $x \in \overline{A \cap B}$ . Hence  $x \notin A \cap B$ . So  $x \notin A$  and  $x \notin B$ . Therefore,  $x \in \overline{A}$  and  $x \in \overline{B}$ . Hence  $x \in \overline{A} \cap \overline{B}$ .

- a) The theorem and proof are correct.
- c) The proof is correct, but the theorem is false. d) The theorem is false, and the proof is in error.
- b) The theorem is correct, but the proof is in error.

e) None of the above.

19. Evaluate the proof of the given theorem:

**Theorem:** Let  $a, b, c \in \mathbb{Z}$  be nonzero. If a | bc and  $a \nmid b$  then a | c.

*Proof*: Assume a bc. This implies bc = ax for some  $x \in \mathbb{Z}$ . Since a doesn't divide b, we must have a divides c.

a) The theorem and proof are correct.

b) The theorem is correct, but the proof makes an unwarranted implication.

- c) The theorem is incorrect, but the proof is correct and proves something else.
- d) The theorem is false, and the proof makes an unjustified step.
- e) all of the above
- f) none of the above

20. Evaluate the proof of the following statement:

**Result:** For all  $n \in \mathbb{N}$ ,  $4|(3^{2n}+7)$ .

*Proof.* Let  $S_n$  be the statement  $4|(3^{2n}+7)$ .

 $S_1$  is true, since  $3^2 + 7 = 16$  is divisible by 4.

For  $k \ge 1$ , assume that  $S_k$  is true, so that  $4|(3^{2k}+7)$ . Then  $3^{2k}+7=4\ell$  for some  $\ell \in \mathbb{Z}$ . Now,

$$3^{2(k+1)} + 7 = 9(3^{2k}) + 7$$
  
= 8(3<sup>2k</sup>) + 3<sup>2k</sup> + 7  
= 8(3<sup>2k</sup>) + 4\ell  
= 4(2(3<sup>2k</sup>) + \ell),

so  $4|(3^{2(k+1)}+7)$ , and we see that  $S_{k+1}$  is true. Hence  $S_k$  implies  $S_{k+1}$ . Therefore, by the principle of mathematical induction,  $S_n$  is true for all  $n \in \mathbb{N}$ .

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- a) The theorem is false but the proof is correct.
- b) The proof contains arithmetic mistakes which make it incorrect.
- c) The proof incorrectly assumes what it is trying to prove.
- d) The proof is a correct proof of the stated result.
- e) None of the above.

#### **Essay Section**

- 21. Write out the first seven rows of Pascal's triangle, and use them to expand the binomial  $(x-y)^7$ .
- 22. Prove that  $3^n > n^2$  for all  $n \in \mathbb{N}$ .
- 23. Phone numbers in the United States consist of a 3 digit area code, followed by a 7 digit local number. (For instance, 801-481-2838 and 283-381-3929 would be valid phone numbers.) Repetitions are allowed. Answer the following questions. (You do not need to simplify your answer. You may leave it as a sum, with exponents and/or factorials.) (1) How many possible phone numbers are there? (2) How many phone numbers do not start with 0 and do not start with 911? (3) How many phone numbers have no repetitions? (4) How many phone numbers have exactly one repetition?
- 24. Let A, B, C be sets. Prove that  $A \cap B = A$  if and only if  $A \subseteq B$ .
- 25. Solve one of the following two problems.
- (A): Prove or disprove: There exist integers x, y, z, each greater than 1 and distinct, with  $x^y = y^z$ .
- (B): Prove of disprove: For every real number x, it is not the case that  $x^4 < x < x^2$ .