Math 290 Practice Exam 2 Solutions

Problem 1. False. The smallest value of n for which $2n^2 - 4n + 31$ is not prime is $n = 30$, for which the polynomial gives 1711 which is the product of 29 and 59. But the easiest value of n to find for which the polynomial $2n^2 - 4n + 31$ is not prime is $n = 31$, for then $2(31)^2 - 4(31) + 31$ factors as $(31)(2(31) - 4 + 1)$.

Problem 2. True. The universally quantified statement is $\forall x \in \mathbb{R}, \sim (x^4 < x < x^2)$, which carrying out the negation gives $\forall x \in \mathbb{R},\ x^4 \geq x$ or $x \geq x^2$. [The statement $x^4 < x < x^2$ is logically the same as $x^4 < x$ AND $x < x^2$.] For $x \geq 1$ or $x \leq 0$ we have $x^4 \geq x$, while for $0 < x < 1$, we have $x \geq x^2$.

Problem 3. False. For each entry in a list of length 11 (with repetition permitted), there are 26 choices. By the Multiplication Principle, there are 26^{11} lists of length 11.

Problem 4. False. For the first entry in a list of length 11 (without repetition), there are 26 choices, for the second entry 25 choices, and so on, until the eleventh entry for which there are 16 choices. By the Multiplication Principle there are

$$(26)(25)\cdots(16) = \frac{26!}{15!}$$

lists of length 11 without repetitions.

Problem 5. True. With $x = 7a$ and $b = 3b$, the Binomial Theorem states that

$$(x+y)^{11} = \sum_{k=0}^{11} \binom{11}{k} x^{11-k} y^k.$$

The coefficient of ab^{10} corresponds to $k = 10$. The coefficient of ab^{10} is

$$(7)(3^{10})\binom{11}{10} = (7)(3^{10})(11).$$

Problem 6. False. A proof by induction always starts with a base case.

Problem 7. False. By stating "The integer 8 works" only gives existence, not uniqueness.

Problem 8. True. The definition of $\binom{n}{k}$ is the number of subsets having k elements taken from a set of n elements.

Problem 9. False. The inclusion $\mathcal{P}(A \cup B) \subset \mathcal{P}(A) \cup \mathcal{P}(B)$ fails when A and B are nonempty sets satisfying there is $a \in A - B$ and there is $b \in B - A$, for then $\{a, b\} \in \mathcal{P}(A \cup B)$, but $\{a, b\} \notin \mathcal{P}(A)$ and $\{a, b\} \notin \mathcal{P}(B)$.

Problem 10. False. In proving $A \subset B$ one starts with $x \in A$ and proceeds to prove that $x \in B$.

Problem 11. (d) The number of lists of length 5 with entries chosen from six symbols is $(6)(5)(4)(3)(2) = 720$.

Problem 12. (d) The number of lists of length 4 with entries chosen from five symbols D, O, N, U, T with repetition allowed, is $5^4 = 625$. The number of lists of length 4 without

O as an entry is $4^4 = 256$. So the number of lists of length 4 (with repetition allowed) that contain at least one O is $625 - 256 = 369$.

Problem 13. (d) We have

$$\binom{10}{5} = \frac{10!}{5!5!} = \frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)(1)} = (2)(9)(2)(7) = (18)(14) = 252.$$

Problem 14. (g) With the second and fourth entries required to be 1, that leaves 6 entries to chosen. For each of these 6 entries there are 2 choices. So there are 2^6 lists of length 8 with the second and fourth entries being 1.

Problem 15. (f) Both (b) and (d) are correct, but (a) assumes what you want to prove, and the assumption in (c) shouldn't lead to a contradiction in general, so (f) is the best answer.

Problem 16. (d) The negation of $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$, $y^2 > x$ is $\exists x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$, $y^2 \leq x$.

Problem 17. (e) We have $2(1) + 3(1) = 5$, $2(2) + 3(1) = 7$, $2(1) + 3(2) = 8$, and $2(3) + 3(1) = 9$. It is easy to get all other bigger numbers.

Problem 18. (d) A counterexample is $A = \{1, 2\}$, $B = \{2, 3\}$ are subsets of the universal set $U = \{1, 2, 3, 4, 5\}$. Here $\overline{A \cap B} = \overline{\{2\}} = \{1, 3, 4, 5\}$ while $\overline{A} \cap \overline{B} = \{3, 4, 5\} \cap \{1, 4, 5\} = \{4, 5\}$, and hence $\overline{A \cap B} \not\subset \overline{A} \cap \overline{B}$.

Problem 19. (d) A counterexample is $a = 6$, $b = 8$, and $c = 15$. For then a divides $bc = 120$ and $a \nmid b$, but $a \nmid c$.

Problem 20. (d) There are no errors in the proof by induction for the statement given.

Problem 21. By the Binomial Theorem we have

$$\begin{aligned}(a+b)^7 &= \sum_{k=1}^{7} \binom{7}{k} a^{n-k} b^k \\ &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.\end{aligned}$$

With $a = x$ and $b = -y$ we have

$$(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7.$$

Problem 22. For $n = 1$ we have $3^1 = 3 > 1^2 = 1$, and for $n = 2$ we have $3^2 = 9 > 4 = 2^2$ (we see why in a moment we needed to check $n = 2$).

Now suppose that $3^k > k^2$ for some $k \geq 2$.

Then

$$3^{k+1} = 3(3^k) > 3(k^2) = k^2 + k^2 + k^2.$$

With $k \geq 2$ we have $k^2 = k(k) \geq 2k$, and with $k \geq 2$ we have $k^2 \geq 1$. Thus

$$3^{k+1} > k^2 + 2k + 1 = (k+1)^2.$$

Therefore by induction $3^n > n^2$ for all $n \geq 1$.

Problem 23. (1) The answer is 10^{10} because there are 10 choices for each of the 10 entries.

(2) The number of possible phone numbers that start with 0 is 10^9 and the number of possible phone numbers starting with 911 is 10^7. So the number of phone numbers that do not start with 0 and do not start with 911 is $10^{10} - 10^9 - 10^7$.

(3) The number of phone numbers with no repetition are 10! because there are 10 choices for the first entry, 9 choices for the second entry, on to 1 choice for the last entry.

(4) The two entries that will have the repeated digit can be chosen in $\binom{10}{2}$ ways. Now consider the two entries as being one entry, giving 9 entries to fill without repetition (in which order matters). There are $10!/1! = 10!$ ways to do this. Thus there are $\binom{10}{2}(10!) = 45(10!)$ phone numbers with exactly one repeated digit.

Problem 24. Suppose $A \cap B = A$. We are to show that $A \subseteq B$. Let $x \in A$. Since $A = A \cap B$ then $x \in A \cap B$. Hence $x \in B$.

Now suppose $A \subseteq B$. We are to show that $A \cap B = A$. Let $x \in A \cap B$. Then $a \in A$ so that $A \cap B \subseteq A$. Now let $x \in A$. Since $A \subseteq B$ we have $x \in B$. Hence $x \in A \cap B$, and so $A \subseteq A \cap B$.

Problem 25. (a) Prove that there exist integers x, y, z, each greater than 1 and distinct, with $x^y = y^z$. The integers $x = 8$, $y = 2$, and $z = 6$ are each greater than 1 and distinct and satisfy $8^2 = 2^6$ (both equal to 64).

(b) Prove that for every real number x, it is not the case that $x^4 < x < x^2$. Let $P(x)$ be the statement $x^4 < x < x^2$. Then $P(x)$ is $x^4 < x$ and $x < x^2$. We are to prove that $\forall x \in \mathbb{R}, \sim P(x)$. Here $\sim P(x)$ is $x^4 \geq x$ or $x \geq x^2$. Now for $x \geq 1$ we have $x^4 \geq x$, for $x \geq 0$ we have $x^4 \geq x$, and for $0 < x < 1$ we have $x \geq x^2$. Thus $\forall x \in \mathbb{R}, \sim P(x)$ is true.