Math 341 Lecture #1
§1.1-1.2: Irrationality of $\sqrt{2}$

1.1 Discussion: The Irrationality of $\sqrt{2}$. In 1940, G.H. Hardy said "Real mathematics must be justified as an art if it can be justified at all."

Showing the irrationality of $\sqrt{2}$ is an illustration of this "art."

Theorem 1.1.1. There is no rational number whose square is 2.

Proof. We will argue by contradiction: suppose there is a rational number p/q for integers p and $q \neq 0$ such that

$$\left(\frac{p}{q}\right)^2 = 2.$$

We may assume in addition that p and q have no common factor.

Now $(p/q)^2 = 2$ rewritten is

$$p^2 = 2q^2.$$

This implies that the integer p^2 is even, and hence p must be even as well because the square of an odd number is odd.

We can thus write $p = 2r$ for an integer r.

Substitution of $p = 2r$ into $p^2 = 2q^2$ yields

$$\begin{aligned}(2r)^2 &= 2q^2\\ 4r^2 &= 2q^2\\ 2r^2 &= q^2.\end{aligned}$$

This implies q is even.

We have shown that both p and q are even integers, they had 2 as a common factor.

But this can not be since we assumed that p and q had no common factors.

This contradiction shows that our original assumption about the existence of a rational number whose square is 2 must be false. □

The Problem with the Rational Numbers. Theorem 1.1.1 hints at what is lacking in the set of rational numbers $\mathbb{Q}$.

The number $\sqrt{2}$ represents a "gap" in $\mathbb{Q}$.

The actual construction of $\mathbb{R}$ from $\mathbb{Q}$ is rather quite complicated and we will not fully investigate this in class (but see Section 8.6 on Dedekind cuts).