Math 341 Lecture #6
§1.6: Cantor's Theorem

We give a less direct proof that $\mathbb{R}$ is uncountable by showing that its subset $(0,1)$ is uncountable.

Before we do so, we recall some facts about decimal expansions of real numbers.

Every irrational number has a nonrepeating decimal expansion that is unique:

$$\sqrt{2} = 1.414\ldots.$$

Every rational number has a repeating decimal expansion:

$$\frac{1}{5} = 0.2000\ldots.$$

Some rational numbers have two repeating decimal expansions:

$$\frac{1}{5} = 0.1999\ldots.$$

How do we know that this second decimal expansion equals $1/5$?

Well, we make use of the convergent geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

where $a \neq 0$ and $|r| < 1$.

Since

$$\begin{aligned}
0.1999\ldots &= 0.1 + 0.0999\ldots \\
&= 0.1 + \frac{1}{100}\big(9.99\ldots\big) \\
&= 0.1 + \frac{1}{100}\left(\frac{9}{10^0} + \frac{9}{10^1} + \frac{9}{10^2}\cdots\right) \\
&= 0.1 + \frac{1}{100}\sum_{n=0}^{\infty} 9\left(\frac{1}{10}\right)^n \\
&= 0.1 + \frac{1}{100}\left(\frac{9}{1-1/10}\right) \\
&= 0.1 + \frac{1}{100}\left(\frac{9}{9/10}\right) \\
&= 0.1 + \frac{1}{100}\left(\frac{10}{9/9}\right) \\
&= 0.1 + \frac{10}{100} \\
&= 0.1 + 0.1 \\
&= 0.2.
\end{aligned}$$

Fact: a rational number a/b in lowest terms has two decimal expansions if and only if the only primes that divide b are 2 or 5.

If a rational number has two decimal expansions, as does $1/5$, the one expansion will repeat 0 from some point, while the other expansion will repeat 9 from some point.

Theorem 1.6.1. The open interval $(0, 1)$ in $\mathbb{R}$ is uncountable.

Proof. We argue by contradiction: suppose there is a bijection $f : \mathbb{N} \to (0, 1)$.

This means that each $x \in (0, 1)$ is the image $x = f(n)$ of a unique $n \in \mathbb{N}$.

Each $a_n = f(n) \in (0, 1)$ has a decimal expansion

$$a_n = 0.a_{n1}a_{n2}a_{n3}\dots$$

were a_{nj} belongs to set of digits $\{0, 1, 2, \dots, 8, 9\}$.

If a_n is irrational then its decimal expansion is unique.

If a_n is rational, its decimal expansion may be unique; if it is not unique then WLOG we assume that the digit 0 repeats from some point on.

We list the images of f starting with $f(1)$, then $f(2)$, etc.:

$$\begin{aligned} a_1 &= f(1) = 0.a_{11}a_{12}a_{13}\dots, \\ a_2 &= f(2) = 0.a_{21}a_{22}a_{23}\dots, \\ a_3 &= f(3) = 0.a_{31}a_{32}a_{33}\dots, \\ &\vdots \end{aligned}$$

Is every real number between $(0, 1)$ really in this list?

We define the number $b = 0.b_1b_2b_3\dots$ by

$$b_i = \begin{cases} 4 & \text{if } a_{ii} = 5, \\ 5 & \text{if } a_{ii} \neq 5. \end{cases}$$

[This differs from the book which uses 2 and 3 instead of 4 and 5.]

By this choice of digits, the decimal expansion for b never has repeating 9's in it, and so b is not an alternative expansion of rational number.

By the choice of b_1 we have that $a_1 \neq b$; by the choice of b_2 we have $a_2 \neq b$; by the choice of b_3 we have $a_3 \neq b$, and on it goes, so that $a_i \neq b$ for all $i \in \mathbb{N}$.

Thus $b \notin f(\mathbb{N})$, and hence f is not surjective. □

Recall that the *power set* of a set A is the collection of all subsets of A, and is denoted by $\mathcal{P}(A)$.

Theorem 1.6.2 (Cantor). For any set A, there does not exist a surjection $f : A \to \mathcal{P}(A)$.

Proof. Suppose there is a surjection $f : A \to \mathcal{P}(A)$.

Thus for each $a \in A$ we get an element $f(a)$ of $\mathcal{P}(A)$, i.e., $f(a)$ is a subset of A.

We will achieve a contradiction by exhibiting a subset B of A such that $B \neq f(a)$ for all $a \in A$.

For each $a \in A$ the subset $f(a)$ of A has either $a \in f(a)$ or $a \notin f(a)$.

Consider the set

$$B = \{a \in A : a \notin f(a)\}.$$

Since by assumption the function f is a surjection, there is $a' \in A$ such that $B = f(a')$.

When we consider the element a' and the set B, we have two possibilities: $a' \in B$ or $a' \notin B$.

If $a' \in B$, then $a' \notin f(a') = B$, a contradiction.

If $a' \notin B$, then $a' \in f(a') = B$, a contradiction.

In both cases we have a contradiction, and so there is no surjection from A to $\mathcal{P}(A)$. $\square$

Corollary. The sets $\mathcal{N}$ and $\mathcal{P}(\mathbb{N})$ do not have the same cardinality.

That raises a question: what other sets has the same cardinality as that of $\mathcal{P}(\mathbb{N})$?

The answer is $(0, 1)$, $\mathbb{R}$, etc.