

Math 341 Lecture #8
§2.4: The Monotone Convergence Theorem
and a First Look at Infinite Series

Not all bounded sequences, like $(-1)^n$, converge, but if we knew the bounded sequence was monotone, then this would change.

Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2 (Monotone Convergence Theorem). If a sequence is monotone and bounded, then it converges.

Proof. Suppose (a_n) is monotone and bounded.

To show convergence by the ϵ - N definition, we will need to “guess” the limit s .

We will assume the sequence is increasing (the decreasing case is similar).

The sequence (a_n) gives a set of points $\{a_n : n \in \mathbb{N}\}$ which by hypothesis is bounded.

As (a_n) is increasing, we take as our “guess” for the limit,

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$

Now we let $\epsilon > 0$ and seek for N .

Because s is the least upper bound of $\{a_n : n \in \mathbb{N}\}$, the number $s - \epsilon$ is not an upper bound, and so there is a number a_N in the sequence for which

$$s - \epsilon < a_N.$$

That is, we have found an N that goes with ϵ .

Since (a_n) is increasing, we know that $a_N \leq a_n$ for all $n \geq N$.

With s being an upper bound we can obtain

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon \quad \text{for all } n \geq N.$$

This of course is nothing more than $|a_n - s| < \epsilon$ for all $n \geq N$. □

The Monotone Convergence Theorem asserts the convergence of a sequence without knowing what the limit is!

There are some instances, depending on how the monotone sequence is defined, that we can get the limit after we use the Monotone Convergence Theorem.

Example. Recall the sequence (x_n) defined inductively by

$$x_1 = 1, \quad x_{n+1} = (1/2)x_n + 1, \quad n \in \mathbb{N}.$$

One uses induction to show that (x_n) is increasing: $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

One also uses induction to show that $x_n \leq 2$ for all $n \in \mathbb{N}$.

Thus (x_n) is monotone and bounded.

By the Monotone Convergence Theorem, (x_n) converges, to a number, say s .

Can we find what it converges to?

Well if $s = \lim x_n$ then does $\lim x_{n+1}$ exist? what is it?

The “new” sequence (x_{n+1}) is almost the “old” sequence (x_n) except it is missing the first term of (x_n) .

So (x_{n+1}) converges to the same thing as (x_n) :

$$\lim_{n \rightarrow \infty} x_{n+1} = s.$$

Now by the Algebraic Limit Theorem, we have

$$s = \lim_{n \rightarrow \infty} x_{n+1} = (1/2) \lim_{n \rightarrow \infty} x_n + 1 = (1/2)s + 1,$$

where we thought of the constant 1 as the constant sequence of 1's.

We solve $s = (1/2)s + 1$ for s to get the limit of the sequence:

$$s = 2.$$

The Monotone Convergence Theorem is extremely useful in the study of infinite series.

Definition 2.4.3. For a sequence (b_n) , an *infinite series* is

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots.$$

The *sequence of partial sums* of an infinite series is

$$s_m = \sum_{n=1}^m b_n = b_1 + b_2 + \cdots + b_m.$$

The infinite series $\sum_{n=1}^{\infty} b_n$ is said to *converge* to B if the sequence of partial sums (s_m) converges to B , and we write

$$B = \sum_{n=1}^{\infty} b_n.$$

Otherwise, if (s_m) diverges, then the infinite series *diverges* as well.

Example 2.4.4. The partial sums for the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

are

$$\begin{aligned} s_m &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} \\ &< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 + 1 - (1/m) < 2. \end{aligned}$$

Thus (s_m) is bounded, and since $1/n^2 > 0$, it is increasing too.

By the Monotone Convergence Theorem, (s_m) converges, and so $\sum_{n=1}^{\infty} 1/n^2$ converges too.

Example 2.4.5 (Harmonic Series). For the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the 2^k term, $k \in \mathbb{N}$, in the sequence of partial sums (s_m) is

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{k}{2}. \end{aligned}$$

This implies that (s_m) is an unbounded increasing sequence, and so it diverges (“converges” to infinity).

Thus the harmonic series diverges.

The argument used in proving the harmonic series diverges can be cast as a general argument.

Theorem 2.4.6 (Cauchy Condensation Test). Suppose (b_n) is decreasing with $b_n \geq 0$ for all $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} \cdots$$

converges.

The proof of this is in the Appendix of this lecture note.

Corollary 2.4.7. The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

The proof of the Corollary requires a few basic facts about geometric series, and is left for you.

Appendix. Proof of Theorem 2.4.6.

Suppose that $\sum_{n=1}^{\infty} 2^n b_{2^n}$ converges.

So its sequence of partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

converges, and hence is bounded; there is $M > 0$ such that $t_k \leq M$ for all $k \in \mathbb{N}$.

Now because $b_n \geq 0$, the sequence of partial sums (s_m) for $\sum_{n=1}^{\infty} b_n$, is increasing.

To show that $\sum_{n=1}^{\infty} b_n$ converges we need only show that (s_m) is bounded.

Fix m and let k be large enough to ensure that

$$m \leq 2^{k+1} - 1.$$

Then

$$s_m \leq s_{2^{k+1}-1}$$

and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} \\ &= t_k. \end{aligned}$$

Thus $s_m \leq t_k \leq M$, and so (s_m) is bounded.

By the Monotone Convergence Theorem, (s_m) converges, and then so does $\sum_{n=1}^{\infty} b_n$.

The opposite direction is shown by its contrapositive: if $\sum_{n=1}^{\infty} 2^n b_{2^n}$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

This argument is very much like that used for the harmonic series in Example 2.4.5. $\square$