Math 341 Lecture #11
§2.6: The Cauchy Criterion

There is way to describe a convergence sequence without an explicit reference to its limit.

This involves comparing the terms of a sequence with themselves!

Definition 2.6.1. A real sequence (a_n) is called *Cauchy* if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ we have

$$|a_n - a_m| < \epsilon.$$

This definition asserts for a sequence that after a certain point (the value of N) the terms in the sequence are all closer to each other than the given value of ϵ.

We will see how this notion of a Cauchy sequence ties in with a convergent sequence.

Theorem 2.6.2. Every convergent sequence is a Cauchy sequence.

Proof. Suppose (x_n) is a convergent sequence with limit x.

For $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$.

We introduce x_m by $|x_n - x_m|$ and use the triangle inequality:

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $m, n \geq N$.

Thus the convergent (x_n) is Cauchy. □

To get the converse – every Cauchy sequence is convergent – takes more work because we have to guess what the limit should be.

Lemma 2.6.3. Cauchy sequences are bounded.

Proof. Suppose (x_n) is Cauchy.

For $\epsilon = 1$ there is $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $m, n \geq N$.

Pick $m = N$, so that $|x_n - x_N| < 1$ for all $n \geq N$.

Using the inequality $\big|\, |a| - |b| \,\big| \leq |a - b|$ (see Exercise 1.2.6(d)), we obtain for $n \geq N$ that

$$\big|\, |x_n| - |x_N| \,\big| \leq |x_n - x_N| < 1.$$

This implies that $|x_n| < |x_N| + 1$ for all $n \geq N$.

That leaves finitely many terms of the sequence at the beginning, and so

$$M = \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for (x_n). □

Theorem 2.6.4 (Cauchy Criterion). A real sequence converges if and only if it is a Cauchy sequence.

Proof. We established that a convergent sequence is Cauchy in Theorem 2.6.2.

So it remains to show that a Cauchy sequence is convergent.

The key to showing this is to find a guess for what the limit is, and we will use the Bolzano-Weierstrass Theorem to do this.

Suppose that (x_n) is Cauchy.

Lemma 2.6.3 guarantees that (x_n) is bounded, and so by the Bolzano-Weierstrass Theorem, the Cauchy sequence (x_n) has a convergent subsequence (x_{n_k}).

Let x be the limit of this convergence subsequence.

The idea now is to show that the original Cauchy sequence (x_n) converges to x as well.

Here is where we will use (again) that (x_n) is Cauchy.

For $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon/2$ for all $m, n \geq N$.

Because (x_{n_k}) converges to x, there is a term in this subsequence, call it x_{n_K} where $n_K \geq N$, such that

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

By this choice of $n_K \geq N$ we obtain

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

for all $n \geq N$.

Therefore, the Cauchy sequence (x_n) is convergent. □

Completeness Revisited. Let's review the logical consequences of the Axiom of Completeness (AoC), that any nonempty bounded above set of real numbers has a least upper bound.

In the proof of the Nested Interval Property (NIP) and the Monotone Convergence Theorem (MCT), we used the Axiom of Completeness.

In the proof of the Bolzano-Weierstrass Theorem (BWT), we used the Nested Interval Property.

And finally, in the proof of the Cauchy Criterion (CC), we used the Bolzano-Weierstrass Theorem.

Are these many theorems logically equivalent?

The NIP implies AoC if we assume that $(1/2)^n \to 0$ (something equivalent to the Archimedean Property) holds.

It is true that, with the assumption of the Archimedean Property, any one of these five – AoC, NIP, MCT, BWT, CC – logically imply the rest.

We know that through Dedekind cuts of rational numbers, that the AoC is not just an axiom but a theorem, and that the Archimedean Property is a consequence of the AoC.

Each one of five – AoC, NIP, MCT, BWT, CC – expresses the idea that there are no "holes" or "gaps" in the real numbers, with each one asserting the completeness of $\mathbb{R}$ in its own way.

With all of these logically equivalent statements about the completeness of $\mathbb{R}$, how then do we *define* the set of real numbers?

For a mind expanding experience, try this: say two *rational* Cauchy sequences (x_n) and (y_n) are equivalent if $x_n - y_n \to 0$; this is an equivalence relation on the set of rational Cauchy sequences; define a real number as an equivalent class of rational Cauchy sequences.

So when we add two real numbers we are really adding two equivalence classes of rational Cauchy sequences!