Math 341 Lecture #15
§3.2: Open and Closed Sets, Part II

Closed Sets. We develop the concepts needed to define what a closed subset of $\mathbb{R}$ is.

Definition 3.2.4. A point $x \in \mathbb{R}$ is a *limit point* of a nonempty $A \subseteq \mathbb{R}$ if every $\epsilon > 0$ we have $(A \cap V_\epsilon(x)) - \{x\} \neq \emptyset$, i.e, $V_\epsilon(x)$ intersects A in some point other than x.

Examples. The endpoint $x = 1$ of the $A = (0, 1]$ is a limit point because every $V_\epsilon(1)$ contains points of $(0, 1]$ other than 1.

The point $x = 1/3$ is a limit point of $A = (0, 1]$ because every $V_\epsilon(1/3)$ contains points of A other than 1/3.

The endpoint 0 of $A = (0, 1]$ is not in A but is a limit point because $(A \cap V_\epsilon(0)) - \{0\} \neq \emptyset$ for every $\epsilon > 0$.

However the point $-1/4$ is not a limit point of $A = (0, 1]$ because not every $V_\epsilon(-1/4)$ contains points of A.

Theorem 3.2.5. A point x is a limit point of a nonempty subset A of $\mathbb{R}$ if and only if $x = \lim a_n$ for some sequence (a_n) contained in A with $a_n \neq x$ for all $n \in \mathbb{N}$.

Proof. Suppose that x is a limit point of A.

For $n \in \mathbb{N}$, take $\epsilon = 1/n$.

With x being a limit point of A, there is a point $a_n \in A$ that is in $V_\epsilon(x)$ with $a_n \neq x$.

To see that (a_n) converges to x, for $\epsilon > 0$, we pick $N \in \mathbb{N}$ such that $1/N < \epsilon$.

Then for all $n \geq N$, we have $a_n \in V_{1/n}(x) \subseteq V_{1/N}(x) \subseteq V_\epsilon(x)$, i.e., $|a_n - x| < 1/N < \epsilon$.

Now suppose there is a sequence (a_n) in A with $a_n \neq x$ for all n, such that $\lim a_n = x$.

Then for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ there holds $a_n \in V_\epsilon(x)$.

In particular, we have for every $\epsilon > 0$ there existence of $a_N \in A$ such that $a_N \in (A \cap V_\epsilon(x)) - \{x\}$.

This says that x is a limit point of A. □

Note that this idea of a limit point x excludes the use of a constant sequence $a_n = x$.

Definition 3.2.6. A point $a \in A$ is an *isolated point of* A if it is not a limit point of A.

Example. Each element of a nonempty finite subset A of $\mathbb{R}$ is an isolated point of A.

However, a nonempty finite subset A of $\mathbb{R}$ does not have any limit points. Why?

Keep in mind that an isolated point of A is an element of A whereas a limit point of A need not be an element of A.

Definition 3.2.7. A set $F \subseteq \mathbb{R}$ is *closed* if F contains all of its limit points.

Theorem 3.2.8. A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has its limit in F also.

This a homework problem 3.2.5.

Example 3.2.9. (i) Does the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

have isolated points? Is it closed?

Each point of A is isolated because for $\epsilon = 1/n - 1/(n+1)$ we have $V_\epsilon(1/n) \cap A = \{1/n\}$, and so $1/n$ is not a limit point of A.

The number 0 is a limit point of A because $1/n \to 0$ where $1/n \neq 0$ for all $n \in \mathbb{N}$.

The set A is not closed because it does not contain its limit point 0.

However, the set $F = A \cup \{0\}$ is closed.

(ii) The closed interval $[c, d]$ for $-\infty < c < d < \infty$ is a closed set.

For a limit point x of $[c, d]$, there is by Theorem 3.2.5 a sequence (x_n) in $[c, d]$ with $x_n \neq x$ and $(x_n) \to x$.

The sequence satisfies $c \leq x_n \leq d$ for all $n \in \mathbb{N}$, so by the Order Limit Theorem we have $c \leq x \leq d$, i.e., $x \in [c, d]$, and so $[c, d]$ is closed.

(iii) The set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$.

Recall Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$) which stated that for every two real numbers $a < b$ there exists a rational number r satisfying $a < r < b$.

Thus for a real number y and $\epsilon = 1/n$ there exists a rational number r_n satisfying $y - 1/n < r_n < y + 1/n$.

If y is irrational then $r_n \neq y$, and if y is rational we choose a rational r_n satisfying $y - 1/n < r_n < y < y + 1/n$.

In either case, we have a rational sequence (r_n) with $r_n \neq y$ such that $r_n \to y$.

Hence by Theorem 3.2.5, the real y is a limit point of $\mathbb{Q}$.

We state this version of the density theorem as its own theorem.

Theorem 3.2.10 (Density of $\mathbb{Q}$ in $\mathbb{R}$). For every $y \in \mathbb{R}$ there exists a sequence of rational numbers converging to y.

Closure. We describe an important topological procedure called closure.

Definition 3.2.11. For a set $A \subseteq \mathbb{R}$ and let L be the set of limit points of A. The **closure** of A is defined to be $\overline{A} = A \cup L$.

We saw in (i) in the previous Example that $\overline{A} = A \cup \{0\}$.

For (ii) we have $\overline{[c, d]} = [c, d]$.

Theorem 3.2.12. For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.

Proof. For a set A, let L be the set of the limit points of A.

Then $\overline{A} = A \cup L$ certainly contains all of the limit points of A.

Is the set $\overline{A}$ closed, i.e., does it contains all of its limit points?

You have it as a homework problem 3.2.7 to supply a proof that $\overline{A}$ is indeed closed.

Now let C be a closed set containing A.

If x is a limit point of A, then there is a sequence (a_n) in A with $a_n \neq x$ for all n, and $a_n \to x$.

Since $A \subseteq C$, we have $a_n \in C$ for all n.

Thus x is a limit point of C, and since C is closed, we have $x \in C$.

This says that $\overline{A} \subseteq C$. □

Complements. If a subset is not open, it is closed? If it is not closed, it is open? The answer to both of these is no, as the half-open, half-closed interval $(0, 1]$ provides a counterexample to both.

However, open and closed are the opposite of each other under complements.

Recall that the complement of a subset A of $\mathbb{R}$ is the set

$$A^c = \{x \in \mathbb{R} : x \notin A\}.$$

Theorem 3.2.13. A set $O \subseteq \mathbb{R}$ is open if and only if O^c is closed, and a set $F \subseteq \mathbb{R}$ is closed if and only if F^c is open.

Proof. Let O be open and let x be a limit point of O^c.

Then every $V_\epsilon(x)$ contains a point of O^c other than x.

If $x \in O$ then as O is open, there is $\epsilon > 0$ such that $V_\epsilon(x) \subseteq O$, contradicting that $V_\epsilon(x)$ contains a point of O^c other than x.

Thus $x \in O^c$, and O^c is closed.

Now assume that O^c is closed, and let $x \in O$.

Then x is not a limit point of O^c, because O^c contains all of its limit points and $x \notin O^c$.

With x not a limit point of O^c there is $\epsilon > 0$ such that $V_\epsilon(x) \cap O^c = \emptyset$, which implies that $V_\epsilon(x) \subseteq O$; thus O is open.

The second part of the theorem follows from the observation that $(E^c)^c = E$: let $O = F^c$ and $O^c = (F^c)^c = F$ and apply the above argument. □

We use Theorems 3.2.3 and 3.2.13 in conjunction with De Morgan's Laws,

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c, \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c,$$

to prove the following.

Theorem 3.2.14. (i) The union of a finite collection of closed sets is closed. (ii) The intersection of an arbitrary collection of closed sets is closed.

The middle-thirds Cantor set is closed because it is the intersection of closed sets.