Math 341 Lecture #16
§3.3: Compact Sets

Some of you might remember that in Math 314 you may learned a little bit about compact sets, that they had something to do with being closed and bounded.

We will put all of this on a rigorous foundation now, using sequences.

Definition 3.3.1. A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K.

Example 3.3.2. A closed interval $[c, d]$ with $-\infty < c < d < \infty$ is a compact set.

The Bolzano-Weierstrass Theorem and the Order Limit Theorem guarantee that any sequence (a_n) with $c \leq a_n \leq d$ for all $n \in \mathbb{N}$ has a convergent subsequence (a_{n_k}) whose limit is in $[c, d]$.

The closed interval $[0, \infty)$ is not compact because the sequence $\{n\}$ in $[0, \infty)$ does not have a convergent subsequence.

What is the difference?

Definition 3.3.3. A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 3.3.4. A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. Let K be compact.

To show that K is bounded, suppose that K is unbounded.

Then for every $n \in \mathbb{N}$ there is $x_n \in K$ such that $|x_n| > n$.

Since K is compact, the sequence (x_n) has a convergent, hence bounded, subsequence (x_{n_j}).

But $|x_{n_j}| > n_j$ with $n_j \to \infty$ as $j \to \infty$, a contradiction.

So K is bounded.

To see that K is closed, we take a limit point x of K and a sequence (x_n) with $x_n \in K$ and $x_n \neq x$ for all $n \in \mathbb{N}$ such that $(x_n) \to x$, and show that $x \in K$.

The compactness of K implies that there is a subsequence (x_{n_j}) that converges to a point that is in K.

Since $(x_n) \to x$, then $(x_{n_j}) \to x$ as well, and so $x \in K$, and K is closed.

Showing that a closed and bounded set is compact is a homework problem 3.3.3. $\square$

We can replace the bounded and closed intervals in the Nested Interval Property with compact sets, and get the same result.

Theorem 3.3.5. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ for compact sets $K_i \subseteq \mathbb{R}$, then $\cap_{n=1}^{\infty} K_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$ pick $x_n \in K_n$.

Because the compact sets are nested, the sequence (x_n) is contained in K_1.

Since K_1 is compact, there is a convergent subsequence (x_{n_j}) with limit $x \in K_1$.

We will show that $x \in K_n$ for all $n \in \mathbb{Z}$, and hence $x \in \cap_{n=1}^{\infty} K_n$.

For each $n_0 \in \mathbb{N}$ we have by the nesting that $x_n \in K_{n_0}$ for all $n \geq n_0$.

Because n_k is a strictly increasing function, there is a choice of k_0 such that for all $k \geq k_0$ we have $x_{n_k} \in K_{n_0}$.

The compactness of K_{n_0} implies that $x \in K_{n_0}$.

Since n_0 is arbitrary, we have that $x \in K_n$ for all $n \in \mathbb{N}$. □

There is another equivalent way to describe (and define) compactness of sets by the use of open sets.

Definition 3.3.6. An *open cover* for $A \subseteq \mathbb{R}$ is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ for which

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.$$

For a given open cover $\{O_\lambda : \lambda \in \Lambda\}$ of A, a *finite subcover* is a finite subcollection of open sets $O_{\lambda_1}, \ldots, O_{\lambda_k}$ in $\{O_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{i=1}^{k} O_{\lambda_i}.$$

Example 3.3.7. For each $x \in (0,1)$, let $\mathcal{O}_x = (x/2, 1)$.

Then the collection $\{\mathcal{O}_x : x \in (0,1)\}$ is an open cover of $(0,1)$ because each $y \in (0,1)$ belongs to O_x for an x satisfying $0 < x/2 < y$.

Does this open cover of $(0,1)$ have a finite subcover?

Suppose there is a finite subcover: there are $x_1, \ldots, x_n \in (0,1)$ such that $O_{x_1}, \ldots, O_{x_n}$ is a finite subcover of $(0,1)$.

For $x' = \min\{x_1, \ldots, x_n\}$, choose $y \in (0, x'/2]$.

Since $O_{x_1}, \ldots, O_{x_n}$ is a finite subcover of $(0,1)$, then

$$y \in \bigcup_{k=1}^{n} O_{x_k}.$$

But $0 < y \leq x_k/2$ for all $k = 1, \ldots, n$, so that $y \notin O_{x_k}$ for all $k = 1, \ldots, n$, and hence

$$y \notin \bigcup_{k=1}^{n} O_{x_k}.$$

Thus the cover $\{O_x : x \in (0,1)\}$ of $(0,1)$ does not have a finite subcover.

Is the set $(0,1)$ compact?

Theorem 3.3.8 (Heine-Borel). For $K \subseteq \mathbb{R}$, the following are equivalent.

(i) K is compact.

(ii) K is closed and bounded.

(iii) Any open cover for K has a finite subcover.

Proof. The equivalence of (i) and (ii) is Theorem 3.3.4.

We will show that (iii) implies (ii).

Suppose that (iii) holds: every open cover of K has a finite subcover.

To show that K is bounded, we consider the open cover $\{V_1(x) : x \in K\}$.

Notice that each $V_1(x)$ has a bounded length of 2.

This open cover has a finite cover: there exist finitely many elements $x_1, \dots, x_k \in K$ such that

$$K \subseteq V_1(x_1) \cup \cdots \cup V_1(x_k).$$

Because the finite cover consists of finitely many open intervals of length 2, the set K must be bounded.

To show that K is closed is more delicate, and it is obtained by contradiction.

Recall Theorem 3.2.8 which states that a set is closed if and only if every Cauchy sequence in the set has its limit in the set as well.

Let (y_n) be a Cauchy sequence in K whose limit $y \notin K$.

Every $x \in K$ is a positive distance away from y, i.e., $\epsilon_x = |x - y|/2 > 0$ for all $x \in K$.

Let $O_x = V_{\epsilon_x}(x)$.

The open cover $\{O_x : x \in K\}$ of K has a finite subcover $O_{x_1}, \dots, O_{x_k}$.

Set

$$\epsilon_0 = \min\{\epsilon_{x_i} : i = 1, \dots, k\}.$$

Then y is at least a distance of $2\epsilon_0$ away from each of $x_1, \dots, x_k$.

Also for this ϵ_0 there is $N \in \mathbb{N}$ such that $|y - y_N| < \epsilon_0$, that is, y_N is within a distance of ϵ_0 of y.

This implies that $y_N \notin O_{x_i}$ for all $i = 1, \dots, k$, and so $y_N \notin \cup_{i=1}^k O_{x_i}$.

But $y_N \in K$ and so y_N is in the finite subcover, a contradiction.

Thus $y \in K$, and K is closed.

Showing that (ii) implies (iii) is left to you to consider (an outline is given in problem 3.3.9 which is not assigned as homework). □