Math 341 Lecture #17
§3.4 Perfect Sets and Connected Sets.

The Cantor set C has another topological property that will prove useful in showing that C is uncountable.

Definition 3.4.1. A set $P \subset \mathbb{R}$ is perfect if it is closed and contains no isolated points.

A finite subset of $\mathbb{R}$ is closed but it is not perfect.

Closed intervals $[c, d]$ with $-\infty < c < d < \infty$, are perfect.

What about the Cantor set?

Theorem 3.4.2. The Cantor set C is perfect.

Proof. Each C_n is a finite union of closed intervals, and so is closed.

Then $C = \cap C_n$ is a closed set.

Now we will show that each $x \in C$ is not isolated by constructing a sequence (x_n) in C with $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.

The closed set C_1 is the union of two closed intervals I_{11} and I_{12} each of length $1/3$.

The point x is in one of these two closed intervals, call it I_1.

The intersection $C_2 \cap I_1$ consists of two closed intervals, one of which contains x.

Pick x_1 to be an endpoint of the other closed interval, and so $x_1 \neq x$.

Because the endpoints of the closed intervals that make C_2 are in C, we have that $x_1 \in C$.

Because x and x_1 are both in I_1 and the length of I_1 is $1/3$, we have $|x_1 - x| \leq 1/3$.

The closed set C_2 is the union of four closed intervals I_{2j}, $j = 1, 2, 3, 4$, each of length $1/9$.

The point x is in one of these four closed intervals, call it I_2.

The intersection $C_3 \cap I_2$ consists of two closed intervals, one of which contains x.

Pick x_2 to be an endpoint of the other closed interval, and so $x_2 \neq x$.

Because x_2 is an endpoint of one of the closed intervals in C_3, we have that $x_2 \in C$.

Because x_2 and x both belong to I_2 which is of length $1/9$, we have $|x_2 - x| \leq 1/9$.

Continuing in this way, we construct a sequence (x_n) in C with $x_n \neq x$ for all $n \in \mathbb{N}$, and $|x_n - x| \leq 1/3^n$.

Thus we have shown that x is a limit point of C, and therefore x is not isolated.

As x was an arbitrary point of C, we have that C is perfect. $\square$

In this proof we used the endpoints of the closed intervals in C_n to form a sequence (x_n) that converged to the given point x in the Cantor set.

Each endpoint is rational of the form $m/3^n$ for $0 \leq m \leq 3^n$, but this does not mean that the limit x of (x_n) is rational.

In fact "most" of the sequences formed from the rational endpoints converge to irrational numbers, and this account for the uncountable nature of the Cantor set.

Theorem 3.4.3. A nonempty perfect set is uncountable.

Proof. A nonempty perfect set P cannot be finite, because in a nonempty finite set each point is isolated.

So a nonempty perfect set is infinite.

Suppose, to the contrary, that P is countable.

Using a bijection $f : \mathbb{N} \to P$ we can enumerate the elements of P as

$$P = \{x_1, x_2, x_3, \ldots\}.$$

The point here is that every element of P appears in this enumerated list.

Let I_1 be a closed interval that contains x_1 in its interior (i.e., x_1 is not an endpoint of I_1).

As P is perfect, the element x_1 is not isolated, so there is some other $y_2 \in P$ such that y_2 is also in the interior of I_1.

Choose a closed interval I_2 centered on y_2 so that $I_2 \subseteq I_1$ and $x_1 \notin I_2$.

Since $y_2 \in I_2$ and $y_2 \in P$ we have $I_2 \cap P \neq \emptyset$.

The element $y_2 \in P$ is not isolated, so there is a $y_3 \in P$ that is in the interior of I_2.

We may choose $y_3 \neq x_2$, for if $y_3 = x_2$ then there will be another choice of $y_3 \in P$ in the interior of I_2 because y_3 is not isolated.

Now choose a closed interval $I_3 \subseteq I_2$ centered on y_3 for which $x_2 \notin I_3$.

Since $y_3 \in I_3$ and $y_3 \in P$ we have $I_3 \cap P \neq \emptyset$.

Carrying out this construction inductively results in a sequence of closed intervals I_n satisfying

(i) $I_{n+1} \subseteq I_n$,

(ii) $x_n \notin I_{n+1}$, and

(iii) $I_n \cap P \neq \emptyset$.

For each $n \in \mathbb{N}$, the set $K_n = I_n \cap P$ is compact because I_n is bounded and $I_n \cap P$ closed.

By Theorem 3.3.5 we have that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

On the other hand, since $K_n \subseteq P$ and $x_n \notin I_{n+1}$ for all $n \in \mathbb{N}$, we have that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

This contradiction implies that P is uncountable. $\square$

Now we turn out attention to another topological notion for subsets of $\mathbb{R}$.

Definition 3.4.4. Two nonempty sets $A, B \subseteq \mathbb{R}$ are *separated* if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

A set $E \subseteq \mathbb{R}$ is *disconnected* if it can be written as $E = A \cup B$ for separated sets A and B.

A set that is not disconnected is called a **connected** set.

Example 3.4.5. (i) The set $A = (1, 2)$ and $B = (2, 5)$ are separated because

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset,\ A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.$$

The set $E = A \cup B$ is disconnected because it is the union of the separated sets A and B.

On the other hand, the sets $C = (1, 2]$ and $D = (2, 5)$ are not separated because $C \cap \overline{D} = \{2\}$.

The set $C \cup D$ is the interval $(1, 5)$ which is connected (although we have not shown this).

(ii) The set of rational numbers is disconnected.

To see this we set

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}),\ B = \mathbb{Q} \cap (\sqrt{2}, \infty).$$

We certainly have $\mathbb{Q} = A \cup B$.

The Order Limit Theorem implies that any limit point of A will be in $(-\infty, \sqrt{2}]$, which is disjoint from B.

Similarly, $A \cap \overline{B} \neq \emptyset$, and so A and B are separated sets.

We conclude that $\mathbb{Q}$ is disconnected.

By carefully working through the logical negations of the quantifiers in the definition of disconnected, we arrive at a positive characterization of connectedness.

Theorem 3.4.6. A set $E \subset \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence (x_n) with all x_n contained in one of A or B, and $x = \lim x_n$ contained in the other.

This notion of connectedness is more relevant in higher dimensions, for in dimension 1, a subset $E \subseteq \mathbb{R}$ is connected precisely when E is an interval.

Theorem 3.4.7. A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows for that $c \in E$ too.

Proof. Suppose that E is connected, let $a, b \in E$, and pick $a < c < b$.

Suppose $c \notin E$, and set

$$A = (-\infty, c) \cap E,\ B = (c, \infty) \cap E.$$

Because $a \in A$ and $b \in B$, both A and B are nonempty; and $E = A \cup B$.

Since any limit point of l of A satisfies $l \le c$ by the Order Limit Theorem, we have that $\overline{A} \cap B = \emptyset$.

Similarly, we have $A \cap \overline{B} = \emptyset$.

Thus A and B are separated set, and so $E = A \cup B$ is disconnected, a contradiction.

Hence, $c \in E$.

Now suppose whenever $a < c < b$ with $a, b \in E$ we have that $c \in E$ too.

We will use Theorem 3.4.6 to show that E is connected.

To this end we write $E = A \cup B$ for nonempty disjoint sets A and B.

Pick $a_0 \in A$ and $b_0 \in B$, and WLOG suppose that $a_0 < b_0$.

Since every $c \in (a_0, b_0)$ must be in E, we have that $I_0 = [a_0, b_0] \subseteq E$.

Bisect I_0 into two equal halves.

The midpoint of I_0 is either in A or B.

If the midpoint of I_0 is in A, take $I_1 = [a_1, b_1]$ to be the right half where a_1 is the midpoint of I_0 and $b_1 = b_0 \in B$.

If the midpoint of I_0 is in B, take $I_1 = [a_1, b_1]$ to be the left half where b_1 is the midpoint of I_0 and $a_1 = a_0 \in A$.

Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$ where $a_n \in A$ and $b_n \in B$, and whose lengths $b_n - a_n$ go to 0 as $n \to \infty$.

By the Nested Interval Property, there exists

$$x \in \bigcap_{n=0}^{\infty} I_n.$$

The sequence (a_n) of left endpoints belongs to A and converges to x, and the sequence (b_n) of right endpoints belongs to B and converges to x as well.

[Note: (a_n) and (b_n) are equivalent Cauchy sequences.]

Since $x \in I_0$ and $I_0 \subseteq E$, we have that $x \in E = A \cup B$, which means that $x \in A$ or $x \in B$.

So there is a limit point of one of A or B that belongs to the other, and by Theorem 3.4.6, the set E is connected. $\square$