Math 341 Lecture #18
§3.5: Baire's Theorem

The middle thirds Cantor set shows us that closed subsets of $\mathbb{R}$ can be deceptively elusive, in contrast to open sets which are unions of open intervals.

This is particularly poignant when we try to assess the "size" of the Cantor set.

We showed that the "length" of the Cantor set is 0 which means that C does not contain any open intervals. (Why?)

We also showed that the Cantor set is perfect and so uncountable (in fact there is a bijection between the Cantor set and the closed interval $[0, 1]$).

In addition to length and cardinality as "sizes" of a set, is a third notion of "size" that we will discuss here.

Recall that arbitrary union of open sets is open, and that an arbitrary intersection of closed sets is closed.

The Cantor set is an example of the latter.

Definition 3.5.1. A set $A \subseteq \mathbb{R}$ is called and F_σ *set* if it can be written as the countable union of closed sets.

Note that a countable union of closed sets is not necessarily closed.

A set $B \subseteq \mathbb{R}$ is called a G_δ *set* if it can be written as the countable intersection of open sets.

Note that a countable intersection of open sets is not necessarily open.

You will show as a homework problem 3.5.1 that the complement of a G_δ set is an F_σ set, and vice versa.

Example. The set of rational numbers $\mathbb{Q}$ is an F_σ set.

We need to find a countable collection of closed subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$.

Since $\mathbb{Q}$ is countable, there is a bijection $f : \mathbb{N} \to \mathbb{Q}$.

For each $n \in \mathbb{N}$, set $F_n = \{f(n)\}$, a closed set with one element.

Then

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} F_n,$$

and hence $\mathbb{Q}$ is an F_σ set.

Passing to the complements, we have that the set $\mathbb{I}$ of irrational numbers is a G_δ set,

$$\mathbb{I} = \bigcap_{n=1}^{\infty} F_n^c,$$

where each $F_n^c = \overline{F}_n^c = (-\infty, f(n)) \cup (f(n), \infty)$ is open and dense.

Recall that a set $G \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if given any two real numbers $a < b$ there exists $c \in G$ such that $a < c < b$.

Another way to think about dense sets is that a $G \subseteq \mathbb{R}$ is dense if and only if $\overline{G} = \mathbb{R}$.

We can do this because we can find sequences in G that converge to any real number precisely when G is dense in $\mathbb{R}$.

Theorem 3.5.2. If $G_1, G_2, G_3, \ldots$ is a countable collection of dense, open sets in $\mathbb{R}$, then

$$\bigcap_{n=1}^{\infty} G_n$$

is a dense subset of $\mathbb{R}$.

Note: This is a slightly different conclusion than that in the text, but this implies the conclusion that $\cap G_n \neq \emptyset$. The proof below is also different than the one outlined in the text.

Proof. To show the denseness of $\cap G_n$ in $\mathbb{R}$ we will show for any open interval $O \subseteq \mathbb{R}$ that there exists a point of $\cap G_n$ contained in O.

Since G_1 is open and dense in $\mathbb{R}$, the intersection $G_1 \cap O$ is a nonempty open set and we can find $x_1 \in G_1$ and $0 < r_1 < 1$ such that

$$\overline{V_{r_1}(x_1)} \subseteq G_1 \cap O.$$

Since G_2 is open and dense in $\mathbb{R}$, the intersection $G_2 \cap V_{r_1}(x_1)$ is a nonempty open set and we can find $x_2 \in G_2$ and $0 < r_2 < 1/2$ such that

$$\overline{V_{r_2}(x_2)} \subseteq G_2 \cap V_{r_1}(x_1).$$

Continuing this process yields sequences of elements $x_n \in G_n$ and radii $0 < r_n < 1/n$ such that

$$\overline{V_{r_{n+1}}(x_{n+1})} \subseteq G_{n+1} \cap V_{r_n}(x_n) \text{ for all } n \in \mathbb{N}.$$

For any $\epsilon > 0$ pick $N \in \mathbb{N}$ so that $2/N < \epsilon$.

For $m, n \geq N$, we have that x_m and x_n both belong to $V_{r_N}(x_N)$ so that

$$|x_n - x_m| < 2r_N < \frac{2}{N} < \epsilon.$$

Thus (x_n) is Cauchy, and so (x_n) converges to some $x \in \mathbb{R}$.

Since each x_i lies in the closed set $\overline{V_{r_{n+1}}(x_{n+1})}$ when $i > n$, it follows that $x \in \overline{V_{r_{n+1}}(x_{n+1})}$.

Since $\overline{V_{r_{n+1}}(x_{n+1})} \subseteq G_{n+1} \cap V_{r_n}(x_n)$ for all $n \in \mathbb{N}$, it follows that $x \in G_{n+1}$ for all $n \in \mathbb{N}$.

Since $\overline{V_{r_1}(x_1)} \subset G_1 \cap O$, we have that $x \in G_1$, so that x is a point of $\cap G_n$.

Also since $\overline{V_{r_1}(x_1)} \subseteq G_1 \cap O$ we have that $x \in O$. $\square$

Opposite to a dense set is another kind of subset of $\mathbb{R}$.

Definition 3.5.3. A set $E \subseteq \mathbb{R}$ is *nowhere dense* if $\overline{E}$ does not contain nonempty open intervals.

Example. The set $\mathbb{Q}$ is dense in $\mathbb{R}$ because every real number is a limit point of $\mathbb{Q}$.

But the set $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ because $\overline{\mathbb{Z}} = \mathbb{Z}$ does not contain nonempty open intervals.

The Cantor set can be shown to be nowhere dense (3.5.9); you have a homework problem to show that a set E is nowhere dense in $\mathbb{R}$ if and only if $\overline{E}^c$ is dense in $\mathbb{R}$ (3.5.8).

Baire's Theorem 3.5.4. The set $\mathbb{R}$ cannot be written as a countable union of nowhere dense sets.

Proof. Suppose to the contrary that there are countable many nowhere dense sets $E_1, E_2, E_3, \ldots$ such that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} E_n.$$

Since $E_n \subseteq \overline{E_n}$ for all n, we have

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n}.$$

But the latter union is a subset of $\mathbb{R}$, so that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} \overline{E_n}.$$

Since each E_n is nowhere dense, each $\overline{E_n}^c$ is an open dense subset of $\mathbb{R}$.

By De Morgan's Law we have

$$\emptyset = \mathbb{R}^c = \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)^c = \bigcap_{n=1}^{\infty} \overline{E_n}^c.$$

But each $\overline{E_n}^c$ is open and dense, so that by Theorem 3.5.2, we have

$$\bigcap_{n=1}^{\infty} \overline{E_n}^c \neq \emptyset.$$

This contradiction shows that $\mathbb{R}$ cannot be written as a countable union of nowhere dense sets. $\square$

The third notion of "size" is that a subset of $\mathbb{R}$ is "thin" or *meager* if it is the countable union of nowhere dense sets.

Subsets of $\mathbb{R}$ that are not meager are considered "fat."

Baire's Theorem says that $\mathbb{R}$ is a "fat" set.

Can $\mathbb{Q}$ be written as a countable union of nowhere dense sets?

Yes, we showed this when we wrote $\mathbb{Q}$ as an F_σ set.

We also showed that $\mathbb{I}$ is a G_δ set, in particular the intersection of open dense subsets, and so $\mathbb{I}$ is dense by Theorem 3.5.2.

The point of Baire's Theorem is that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is much bigger than $\mathbb{Q}$.