Math 341 Lecture #19
§4.1: Examples of Dirichlet and Thomae

We begin a discussion about the continuity of a function $f : A \to \mathbb{R}$, for a nonempty $A \subset \mathbb{R}$.

Recall from Calculus I that we say f is continuous at a point $a \in \mathbb{R}$ if $f(a)$ exists (i.e., $a \in A$), $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x) = f(a)$.

[We are leaving the notion of the limit of a function vague for now; we will see the rigorous definition next time.]

For a function $f : A \to \mathbb{R}$, we let D_f denote the set of points in A where f is not continuous.

What kind of a subset of $\mathbb{R}$ can D_f be?

Example. Dirichlet defined a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For any $c \in \mathbb{R}$ we can find sequences (x_n) in $\mathbb{Q}$ and (y_n) in $\mathbb{Q}^c$ such that $x_n \to c$ and $y_n \to c$, but for which $g(x_n) = 1$ and $g(y_n) = 0$ for all $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} g(x_n) \neq \lim_{n\to\infty} g(y_n).$$

This suggests that f is not continuous at c, and as c was arbitrary, that f is not continuous at any $c \in \mathbb{R}$.

We have that $D_g = \mathbb{R}$.

Example. A modification of Dirichlet's function results in a function that is continuous at just one point.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For a nonzero c we can find sequences (x_n) in $\mathbb{Q}$ and (y_n) in $\mathbb{Q}^c$ such that $x_n \to c$ and $y_n \to c$, but for which $h(x_n) = x_n$ and $h(y_n) = 0$ for all $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} h(x_n) = c \neq 0 = \lim_{n\to\infty} h(y_n).$$

This suggests that the function h is not continuous at any point $c \neq 0$.

However, if $c = 0$, then for any sequence (z_n) in $\mathbb{R}$ with $z_n \to 0$ we have $|h(z_n)| \leq |z_n|$, so that $h(z_n) \to 0$ as well.

Thus h is continuous at $c = 0$.

We have that $D_h = \mathbb{R} - \{0\}$.

Example. Thomae defined a function $t : \mathbb{R} \to \mathbb{R}$ by

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ in lowest terms with } n > 0, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For $c \in \mathbb{Q}$, we have $t(c) > 0$.

For a sequence (y_n) in $\mathbb{Q}^c$ such that $y_n \to c$, we have $t(y_n) = 0$ for all $n \in \mathbb{N}$, so that

$$t(c) \neq 0 = \lim_{n\to\infty} t(y_n).$$

This suggests that t is discontinuous at every rational point.

On the other hand, if c is irrational, we have $t(c) = 0$.

For any sequence (x_n) in $\mathbb{R}$ such that $x_n \to c$ we have $t(x_n) = 0$ when $x_n \notin \mathbb{Q}$ or $t(x_n)$ is the reciprocal of the positive denominator of the rational x_n is lowest terms.

The closer x_n is to the irrational c, the larger the denominator of x_n is, so that $t(x_n)$ is as close to 0 as needed.

The result of this is that $t(x_n) \to 0$ as $n \to \infty$, that is, we have

$$\lim_{n\to\infty} t(x_n) = 0 = t(c),$$

suggesting that t is continuous at every irrational c.

We have that $D_t = \mathbb{Q}$.

Example. Define a function $s : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = [[x]]$$

where $[[x]]$ is the largest integer n such that $n \leq s$.

For $c \in \mathbb{R}$ such that $n < c < n + 1$ for $n \in \mathbb{N}$, we have for any sequence (x_n) converging to c that

$$\lim_{n\to\infty} s(x_n) = n = [[c]].$$

On the other hand, for $c = n$ for $n \in \mathbb{N}$, we take a sequence (y_n) such that $n-1 < y_n < n$ and $y_n \to c$, so that

$$\lim_{n\to\infty} s(y_n) = n - 1 \neq [[c]] = n.$$

This suggests that s is discontinuous at every integer point, and we have that $D_s = \mathbb{Z}$.

Example. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \in \mathbb{Q} \cap (0, 1), \\ 0 & \text{if } x \in (0, 1) - \mathbb{Q}, \\ 0 & \text{if } x \geq 1. \end{cases}$$

The function is continuous at every $c < 0$ and at every $c > 1$.

As with the modified Dirichlet function, this function f is continuous at $c = 0$, but discontinuous at every $c \in (0, 1)$.

This function is also discontinuous at $c = 1$ because for a rational sequence (x_n) in $(0, 1)$ with $x_n \to 1$ we have $f(x_n) = x_n \to 1$, while for any sequence (y_n) with $y_n > 1$ and $y_n \to 1$ we have $f(y_n) \to 0$.

So here we have $D_f = (0, 1]$.

With all of the examples we have explored, what is the topological property shared by the set of discontinuities? Open, closed, compact, connected, F_σ, G_δ?

If you are thinking an F_σ set, you are correct.

To prove this is somewhat involved, so we focus in Section 4.6 on a simpler class of functions f for which D_f is more readily understood.