Math 341 Lecture #20
§4.2: Functional Limits

We will now rigorously define $\lim_{x\to c} f(x)$ for a function $f : A \to \mathbb{R}$ with $\emptyset \neq A \subseteq \mathbb{R}$ (and A not assumed to be an interval).

Recall that a limit point c of A is a point $c \in \mathbb{R}$ such that $(A \cap V_\epsilon(c)) - \{c\} \neq \emptyset$ for all $\epsilon > 0$.

Equivalently, c is a limit point of A if there is a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$.

For a limit point c of A, we remember that

$$\lim_{x\to c} f(x) = L$$

means that as x approaches c, the value of $f(x)$ approaches L.

And you might remember the $\epsilon - \delta$ version of this.

Definition 4.2.1. Let $f : A \to \mathbb{R}$, and let c be a limit point of A (so that c is not necessarily in the nonempty A). We say that

$$\lim_{x\to c} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ with $x \in A$ it follows that $|f(x) - L| < \epsilon$.

We can recast this $\epsilon-\delta$ definition of limit in the topological setting: we say $\lim_{x\to c} f(x) = L$ if for every $V_\epsilon(L)$ there exists $V_\delta(c)$ such that for every $x \in (V_\delta(c) \cap A) - \{c\}$, it follows that $f(x) \in V_\epsilon(L)$.

The appearance of $x \in A$ in these equivalent definitions of a limit is to remind us that x has to be in the domain of f; it isn't always the case the f is defined for all points nearby c.

Example 4.2.2. (i) For $f(x) = 3x + 1$ with domain $A = \mathbb{R}$ we will show that

$$\lim_{x\to 2} f(x) = 7.$$

For each $\epsilon > 0$ we have to find $\delta > 0$ such that $0 < |x - 2| < \delta$ leads to $|f(x) - 7| < \epsilon$.

We look at what f is doing in relation to is alleged limit of 7:

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2|.$$

Since we want this to be smaller than ϵ when $0 < |x - 2| < \delta$, we pick

$$\delta = \frac{\epsilon}{3}.$$

Then we have that

$$|f(x) - 7| = 3|x - 2| < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

(ii) For $g(x) = x^2$ we will show that

$$\lim_{x \to 2} g(x) = 4.$$

We start with how $g(x)$ relates with 4:

$$|g(x) - 4| = |x^2 - 4| = |(x+2)(x-2)| = |x+2|\ |x-2|.$$

The term $|x-2|$ we can control with δ, but what do we do with $|x+2|$?

This is where the flexible to choose δ comes into play.

We are only interested in what happens to $g(x)$ when x is close to 2, and so we choose to keep δ from getting too big.

When $\delta \leq 1$, the inequality $|x-2| < \delta$ implies that $|x+2| < 5$.

To get an ϵ into this we choose $\delta = \min\{1, \epsilon/5\}$ which forces δ to never be bigger than 1.

Then for $0 < |x-2| < \delta$ we have that

$$|x^2 - 4| = |x+2|\ |x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$

We can recast the definition of a functional limit in terms of sequences.

Theorem 4.2.3 (Sequential Criterion for Functional Limits). For a function $f : A \to \mathbb{R}$ and a limit point c of A, the following are equivalent.

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all sequences (x_n) in A satisfying $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$, we have that $f(x_n) \to L$.

Proof. Suppose that $\lim_{x \to c} f(x) = L$.

For $\epsilon > 0$ there is $\delta > 0$ such that $f(x) \in V_\epsilon(L)$ whenever $x \in (V_\delta(c) \cap A) - \{c\}$.

Consider an arbitrary sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$.

Because $x_n \to c$, there is $N \in \mathbb{N}$ such that $x_n \in (V_\delta(c) \cap A) - \{c\}$ for all $n \geq N$.

Having $x_n \in (V_\delta(c) \cap A) - \{c\}$ for all $n \geq N$ implies that $f(x_n) \in V_\epsilon(L)$ for all $n \geq N$.

This says precisely that $f(x_n) \to L$.

We will argue the other direction by contradiction.

We assume that for all sequences (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$ we have $f(x_n) \to L$, but that $\lim_{x \to c} f(x) \neq L$.

The latter means that there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists $x \in (V_\delta(c) \cap A) - \{c\}$ such that $f(x) \notin V_{\epsilon_0}(L)$.

We use this to construct a sequence that will give a contradiction.

For each $n \in \mathbb{N}$ we set $\delta_n = 1/n$ and choose $x_n \in (V_{\delta_n}(c) \cap A) - \{c\}$ for which $f(x_n) \notin V_{\epsilon_0}(L)$.

The sequence (x_n) converges to c, but $f(x_n) \not\to L$, a contradiction. □

Now we can apply the theory of sequences to derive familiar results about functional limits.

Corollary 4.2.4 (The Algebraic Limit Theorem for Functional Limits). Let f and g be real-valued functions defined on $A \subseteq \mathbb{R}$, and assume that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then

(i) $\lim\limits_{x\to c} kf(x) = kL$ for all $k \in \mathbb{R}$,

(ii) $\lim\limits_{x\to c} \big[f(x) + g(x)\big] = L + M$,

(iii) $\lim\limits_{x\to c} \big[f(x)g(x)\big] = LM$, and

(iv) $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$.

The proof of these is a simple consequence of the Algebraic Limit Theorem for sequences.

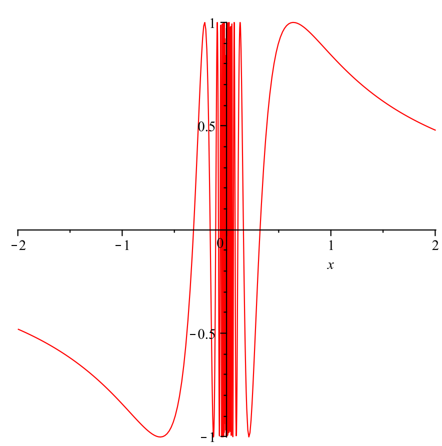
Corollary 4.2.5 (Divergence Criterion for Functional Limits). Let $f : A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$ and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$, $y_n \neq c$ for all $n \in \mathbb{N}$, and $x_n \to c$ and $y_n \to c$, and

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

then $\lim_{x\to c} f(x)$ does not exist.

You should be able to see why this Corollary is true.

Example 4.2.6. Does the function $f(x) = \sin(1/x)$ defined on $A = \mathbb{R} \setminus \{0\}$ have a limit as $x \to 0$?



For $n \in \mathbb{N}$, if

$$x_n = \frac{1}{2n\pi}, \ y_n = \frac{1}{2n\pi + \pi/2}$$

then $x_n \to 0$ (with $x_n \neq 0$) and $y_n \to 0$ (with $y_n \neq 0$), and $f(x_n) = \sin(2n\pi) = 0$ and $f(y_n) = f(2n\pi + \pi/2) = 1$ for all $n \in \mathbb{N}$, so that $f(x_n) \to 0$ while $f(y_n) \to 1$.

By the Divergence Criterion for Functional Limits, we have that $\lim_{x\to 0} f(x)$ does not exist.