Math 341 Lecture #22
§4.4: Continuous Functions on Compact Sets

The Extreme Value Theorem. The topological terms of open, closed, bounded, compact, perfect, and connected are all used to describe subsets of $\mathbb{R}$.

A function $f : A \to \mathbb{R}$ maps a subset A of $\mathbb{R}$ to a subset $f(A)$ of $\mathbb{R}$.

If f is continuous and $B \subseteq A$ has topological property X, does $f(B)$ have topological property X as well?

Examples. (a) If B is open, then is $f(B)$ open too?

The function $f(x) = x^2$ is continuous on $A = \mathbb{R}$, the set $B = (-1, 1) \subseteq$ is open, but $f(B) = [0, 1)$ is not open.

So continuous functions do not in general take open sets to open sets.

(b) If B is closed, then is $f(B)$ closed?

The function

$$f(x) = \frac{1}{1 + x^2}$$

is continuous on $A = \mathbb{R}$, the set $B = [0, \infty) \subseteq A$ is closed, but $f(B) = (0, 1]$ is not closed.

So continuous functions do not in general take closed sets to closed sets.

(c) If B is bounded, then is $f(B)$ bounded too?

The function $f(x) = 1/x$ is continuous on $A = \mathbb{R} - \{0\}$, the set $B = (0, 1) \subseteq A$ is bounded, but $f(B) = [1, \infty)$ is not bounded.

So continuous functions do not in general take bounded sets to bounded sets

So what topological property does a continuous map preserve?

Theorem 4.4.1 (Preservation of Compact Sets). If $f : A \to \mathbb{R}$ is continuous and $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof. We will show that a sequence (y_n) in $f(K)$ has a convergent subsequence whose limit is in $f(K)$, and therefore $f(K)$ is compact (by Definition 3.3.1).

With $y_n \in f(K)$ there is (at least one) $x_n \in K$ such that $y_n = f(x_n)$.

This gives a sequence (x_n) in K.

The compactness of K implies that (x_n) has a convergent subsequence (x_{n_k}), whose limit $x \in K$.

Since $(x_{n_k}) \to x$ and f is continuous, we have that $y_{n_k} = f(x_{n_k}) \to f(x)$.

Since $x \in K$, we know that $f(x) \in f(K)$, so that (y_n) has a convergent subsequence (y_{n_k}) whose limit $f(x)$ is in K; hence $f(K)$ is compact. □

Recall the a continuous function defined on a closed interval of finite length, always attains a maximal value and a minimum value.

Have any of you seen a proof of this Math 112 result?

Well, we can now give a proof of this.

Theorem 4.4.2 (The Extreme Value Theorem). If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Proof. Under the assumptions, we have that $f(K)$ is compact by Theorem 4.4.1.

Since $f(K)$ is bounded, the quantity $\alpha = \sup f(K)$ exists by the Axiom of Completeness.

For each $n \in \mathbb{N}$, there is then $y_n \in f(K)$ such that $\alpha - 1/n < y_n \leq \alpha$, and hence $y_n \to \alpha$.

If $y_n < \alpha$ for all n, then α is a limit point of $f(K)$, and since $f(K)$ is closed, we have $\alpha \in f(K)$.

If $y_n = \alpha$ for some n, then $\alpha \in f(K)$.

Thus, in either case, there is $x_1 \in K$ such that $f(x) \leq \alpha = f(x_1)$ for all $x \in K$.

Similarly, there is $x_0 \in f(K)$ such that $f(x_0) \leq f(x)$ for all $x \in K$. □

Uniform Continuity. We turn our attention to another consequence of continuity of a function on a compact set, for which we set the stage.

Example 4.4.3. (i) In showing that the function $f(x) = 3x + 1$ is continuous at an arbitrary $c \in \mathbb{R}$, we choose $\delta > 0$ that corresponds to $\epsilon > 0$ by considering

$$|f(x) - f(c)| = |3x + 1 - 3c - 1| = 3|x - c|.$$

Here the choice of $\delta = \epsilon/3$ is independent of c that gives $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

Changing the value of c does not change the value of δ needed for continuity.

(ii) In showing that $f(x) = x^2$ is continuous at an arbitrary $c \in \mathbb{R}$, we choose δ that corresponds to $\epsilon > 0$ by considering

$$|f(x) - f(c)| = |x^2 - c^2| = |x + c|\ |x - c|$$

where when $|x - c| < \delta \leq 1$ we have $|x| - |c| \leq |x - c| < 1$, so that $|x| < |c| + 1$, and hence

$$|x + c| \leq |x| + |c| \leq (|c| + 1) + |c| = 2|c| + 1.$$

Here the choice of $\delta = \min\{1, \epsilon/(2|c| + 1)\}$ is dependent on c that gives $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

Changing the value of c changes the value of δ needed for continuity.

Definition 4.4.4. A function $f : A \to \mathbb{R}$ is *uniformly continuous on* A if for every $\epsilon > 0$ there exists δ such that $|x - y| < \delta$ for $x, y \in A$ implies that $|f(x) - f(y)| < \epsilon$.

This says that there is one choice of δ needed for continuity of f for every $c \in A$.

The function $f(x) = 3x + 1$ in Example 4.4.3 (i) is uniformly continuous on $A = \mathbb{R}$ because the choice of δ needed for continuity of f at c is independent of c.

The function $f(x) = x^2$ in Example 4.4.3 (ii) is not uniformly continuous on $A = \mathbb{R}$ because the choice of δ needed for continuity of f at c depends on c: the larger $|c|$ is, the smaller δ has to be for continuity, and so there is not one choice of δ that will do the job.

We have a test for detecting the lack of uniform continuity.

Theorem 4.4.5 (Sequential Criterion for Nonuniform Continuity). A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof. The logical negation of the definition of uniform continuity is: there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists $x, y \in A$ satisfying $|x-y| < \delta$ for which $|f(x)-f(y)| \geq \epsilon_0$.

For each $\delta = 1/n$ choose $x_n, y_n \in A$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

Then $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)| \geq \epsilon_0$. □

Example 4.4.6. The function $f(x) = \sin(1/x)$ is continuous on the bounded open $A = (0, 1)$, but it is not uniformly continuous on A.

Keeping Theorem 4.4.5 in mind, for $\epsilon_0 = 2$ and sequences

$$x_n = \frac{1}{2n\pi + \pi/2}, \ y_n = \frac{1}{2n\pi + 3\pi/2}$$

we have $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)| = |1 - (-1)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Is there a topological property on A that would guarantee that a function continuous on A is uniformly continuous on A?

Theorem 4.4.7. A function that is continuous on a compact set K is uniformly continuous on K.

Proof. Suppose for a compact $K \subseteq \mathbb{R}$, that a continuous function $f : K \to \mathbb{R}$ is not uniformly continuous.

By Theorem 4.4.5, there exist $\epsilon_0 > 0$ and sequences (x_n) and (y_n) in K such that $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)| \geq \epsilon_0$.

By the compactness of K the sequence (x_n) has a convergent subsequence x_{n_k} whose limit x is in K.

For the subsequence (y_{n_k}) we have by the Algebraic Limit Theorem that

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} \big((y_{n_k} - x_{n_k}) + x_{n_k}\big) = 0 + x = x.$$

Thus (x_{n_k}) and (y_{n_k}) converge to $x \in K$.

Because f is continuous at every point of K, we have the $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$, which by the Algebraic Limit Theorem implies that

$$\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0.$$

This contradicts that $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0 > 0$. □