Math 341 Lecture #30
§6.4: Series of Functions

Recall that we constructed the continuous nowhere differentiable function from Section 5.4 by using a series.

We will develop the tools necessary to showing that this pointwise convergent series is indeed a continuous function.

Definition 6.4.1. Let f_n, $n \in \mathbb{N}$, and f be functions on $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence of partial sums,

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$.

The series *converges uniformly on* A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.

Uniform convergence of a series on A implies pointwise convergence of the series on A.

For a pointwise or uniformly convergent series we write

$$f = \sum_{n=1}^{\infty} f_n \text{ or } f(x) = \sum_{n=1}^{\infty} f_n(x).$$

When the functions f_n are continuous on A, each partial sum $s_k(x)$ is continuous on A by the Algebraic Continuity Theorem (Theorem 4.3.4).

We can therefore apply the theory for uniformly convergent sequences to series.

Theorem 6.4.2 (Term-by-term Continuity Theorem). Let f_n be continuous functions on $A \subseteq \mathbb{R}$. If

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly to f on A, then f is continuous on A.

Proof. We apply Theorem 6.2.6 to the partial sums $s_k = f_1 + f_2 + \cdots + f_k$. □

When the functions f_n are differentiable on a closed interval $[a, b]$, we have that each partial sum s_k is differentiable on $[a, b]$ as well.

We recall the result on differentiability of the limit function from Section 6.3.

Theorem 6.3.3. Let (f_n) be a sequence of differentiable functions on the closed interval $[a, b]$, and assume (f_n') converges uniformly to a function g on $[a, b]$. If there is a point $x_0 \in [a, b]$ such that $f_n(x_0)$ is a convergent sequence, then (f_n) converges uniformly, and the limit function $f = \lim f_n$ is differentiable with $f' = g$.

We now apply this Theorem to series.

Theorem 6.4.3 (Term-by-term Differentiability Theorem). Let f_n be differentiable functions on a closed interval $[a, b]$. If

$$\sum_{n=1}^{\infty} f_n'$$

converges uniformly to a function g on $[a, b]$, and if there is $x_0 \in [a, b]$ for which

$$\sum_{n=1}^{\infty} f_n(x_0)$$

converges, then

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly to a differentiable function $f(x)$ satisfying $f' = g$ on $[a, b]$. In other words we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ and } f' = \sum_{n=1}^{\infty} f_n'(x).$$

Proof. We apply Theorem 6.3.3 to the partial sums $s_k = f_1 + f_2 + \cdots + f_k$, where we have $s_k' = f_1' + f_2' + \cdots f_k'$. □

Like sequences, we have a Cauchy condition for series which is useful for examples and proofs.

Theorem 6.4.4 (Cauchy Criterion for Uniform Convergence of Series). A series

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon \text{ for all } x \in A.$$

Proof. For the partial sums s_k we have when $n > m$ that for all $x \in A$,

$$|s_n(x) - s_m(x)| = |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)|.$$

We then apply the Cauchy Criterion for Uniform Convergence for sequences (Theorem 6.2.5). □

Uniform convergence is preferred over pointwise convergence because uniform convergence gives better results for the limit functions.

How do we show a series is uniformly convergent?

Corollary 6.4.5 (Weierstrass M-Test). For each $n \in \mathbb{N}$, let f_n be a function on $A \subseteq \mathbb{R}$. If there exists a real $M_n > 0$ such that

$$|f_n(x)| \leq M_n \text{ for all } x \in A,$$

and the series

$$\sum_{n=1}^{\infty} M_n$$

converges, then

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on A.

Proof. Let $\epsilon > 0$.

Since the series $\sum_{n=1}^{\infty} M_n$ converges, there exists by the Cauchy Criterion for Series (Theorem 2.7.2) an $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have that

$$M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon.$$

Since for each $k \in \mathbb{N}$ we have $|f_k(x)| \leq M_k$ for all $x \in A$, it follows for $n > m \geq N$ that

$$\begin{aligned}|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| &\leq |f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)| \\ &\leq M_{m+1} + M_{m+2} + \cdots + M_n \\ &< \epsilon.\end{aligned}$$

Thus by the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.4.4), we conclude that $\sum_{n=1}^{\infty} f_n$ converges uniformly. □

Example. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + x^4}$$

consisting of differentiable functions converges to a differentiable function.

Since for each $n \in \mathbb{N}$ we have $n^4 + x^4 \geq n^4$ for all $x \in \mathbb{R}$, then

$$\frac{n}{n^4 + x^4} \leq \frac{n}{n^4} = \frac{1}{n^3}.$$

Setting $f_n(x) = n/(n^4 + x^4)$ and $M_n = 1/n^3$ we have for each $n \in \mathbb{N}$ that $|f_n(x)| \leq M_n$ for all $x \in \mathbb{R}$.

The series

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by Corollary 2.4.7.

By the Weierstrass M-Test, the series

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{n}{n^4 + x^4}$$

converges uniformly, so that its limit function $f(x)$ is continuous on $\mathbb{R}$ by Theorem 6.4.2. For each $n \in \mathbb{N}$ we have

$$f_n'(x) = -\frac{4nx^3}{(n^4 + x^4)^2}.$$

For a real $K > 0$, we have for all $x \in [-K, K]$ that

$$|f_n'(x)| \leq \left| \frac{4nx^3}{(n^4 + x^4)^2} \right| \leq \frac{4nK^3}{n^8} = \frac{4K^3}{n^7}.$$

Since

$$\sum_{n=1}^{\infty} \frac{4K^3}{n^7}$$

converges, we have that f_n' converges uniformly on $[-K, K]$ by the Weierstrass M-Test. Since $\sum_{n=1}^{\infty} f_n(x_0)$ converges to $f(x_0)$ for any $x_0 \in [-K, K]$, then by Theorem 6.4.3, we have that $f(x)$ is differentiable on $[-K, K]$ with

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{-4nx^3}{(n^4 + x^4)^2}.$$

Since $K > 0$ is arbitrary, we have that f is differentiable on $\mathbb{R}$.