Math 341 Lecture #31
§6.5: Power Series

We now turn our attention to a particular kind of series of functions, namely, power series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

In terms of a series of functions, we have $f_n(x) = a_n x^n$ which is infinitely differentiable (and continuous) for each $n = 0, 1, 2, 3, \ldots$.

Theorem 6.5.1. If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

See the Appendix for a proof.

The main implication of Theorem 6.5.1 is that the subset of $\mathbb{R}$ on which a power series converges is either $\{0\}$, a bounded interval centered at 0, or $\mathbb{R}$.

In the bounded interval case, there is some ambiguity about the endpoints because of the strict inequality in Theorem 6.5.1.

The bounded interval of convergence can be one of four possible forms: $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$ for some $R > 0$.

We call R the *radius of convergence* in the bounded interval case.

We assign $R = 0$ when the set of convergence is $\{0\}$, and $R = \infty$ when the set of convergence is $\mathbb{R}$.

Recall that a standard way to find the radius of convergence R for a power series is the Ratio Test.

We now turn our attention to properties of a power series that converges absolutely at some point.

Theorem 6.5.2. If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then the power series converges uniformly on the compact interval $[-c, c]$ where $c = |x_0|$.

Proof. Suppose for some x_0 we have that

$$\sum_{n=0}^{\infty} |a_n x_0^n|$$

converges.

With $M_n = |a_n x_0^n|$ for each $n = 0, 1, 2, 3, \ldots$, we have for all x satisfying $|x| \leq |x_0|$ that

$$|a_n x^n| \leq |a_n x_0^n| = M_n.$$

Since we have the convergence of

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |a_n x_0^n|,$$

the Weierstrass M-Test implies that

$$\sum_{n=0}^{\infty} a_n x^n$$

converges uniformly on $A = [-c, c]$ for $c = |x_0|$. $\square$

We can now prove that when a power series converges on an open interval $(-R, R)$ with $R > 0$ or $R = \infty$, the power series is a continuous function on $(-R, R)$.

For a fixed x_1 satisfying $0 < x_1 < R$ we pick x_0 such that $x_1 < x_0 < R$.

The convergence of $\sum_{n=0}^{\infty} a_n x_0^n$ implies by Theorem 6.5.1 that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely on the open interval $|x| < x_0$.

Since $0 < x_1 < x_0$, we have that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at x_1.

By Theorem 6.5.2, we have that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-x_1, x_1]$.

Since each $f_n(x) = a_n x^n$ is continuous on $[-x_1, x_1]$, we have by Theorem 6.4.2 that the power series is continuous on $[-x_1, x_1]$.

Since x_1 satisfying $0 < x_1 < R$ is arbitrary, we have that $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.

Example. Determine where the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

is continuous.

Without using the Ratio Test, we can determine the radius of convergence for this series.

The n in the denominator suggests that we evaluate the power series at $x = -1$: we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges by Corollary 2.4.7.

This says that the radius of convergence satisfies $R \leq 1$.

For any $0 < x \leq 1$, we have that x^n/n is a decreasing sequence converging to 0, so by the Alternating Series Test (Theorem 2.7.7), we have convergence of the power series on $(0, 1]$.

This says that the radius of convergence satisfies $R \geq 1$.

Thus we have that $R = 1$, and so the power series is continuous on $(-1, 1)$.

The power series converges at $x = 1$, but it is continuous at $x = 1$? Could the conditional convergence at $x = 1$ prevent continuity at $x = 1$?

We see in this example that the interval of convergence could be of the form $(-R, R]$, and the question is then is the power series continuous on $(-R, R]$.

We answer this in the affirmative.

Theorem 6.5.4 (Abel's Theorem). If $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at $x = R > 0$, then the power series converges uniformly on $[0, R]$. A similar result holds when the power series converges at $x = -R < 0$.

See the Appendix for a proof.

This answers the question that when the interval of convergence of a power series includes or one or both endpoints we have continuity of the power series on that interval.

We can say something even stronger.

Theorem 6.5.5. If a power series converges pointwise on a set $A \subseteq \mathbb{R}$, then it converges uniformly on any compact subset K of A.

Proof. Let K be a compact subset of A.

Then the points $a = \inf\ K$ and $b = \sup\ K$ both belong to K, and we have that $K \subseteq [a, b]$.

Since $a, b \in A$, we have the convergence of the power series at a and b.

If $a < 0$ and $b > 0$, then by Abel's Theorem we have uniform convergence of the power series on $[a, 0]$ and $[0, b]$, hence uniform convergence on $[a, b]$, and thus on K as well.

If $a \geq 0$ then $b \geq a \geq 0$, giving uniform convergence on $[0, b]$ and hence on K.

If $b \leq 0$ then $a \leq b \leq 0$, giving uniform convergence on $[a, 0]$ and hence on K. □

How about the differentiability of a convergent power series?

According to Theorem 6.4.3, we need uniform convergence of the term by term differentiated series.

By Theorem 6.5.5, it suffices to get pointwise convergence of the term by term differentiated series.

Theorem 6.5.6. If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$ for $R > 0$, then the term-wise differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for all $x \in (-R, R)$ as well.

See the Appendix for a proof.

We can now summarize the totality of the theory of convergence power series we have proven in this section.

Theorem 6.5.7. Suppose that

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on an interval $A \subseteq \mathbb{R}$. Then g is continuous on A and differentiable on any open subinterval of A, where

$$g'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, $g(x)$ is infinitely differentiable on any open subinterval of A, and successive derivatives of g are obtained through term by term differentiation of $g'(x)$, $g''(x)$, etc.

Appendix.

Proof of Theorem 6.5.1. Suppose there is $x_0 \in \mathbb{R}$ such that $\sum_{n=0}^{\infty} a_n x_0^n$ converges.

By Theorem 2.7.3 we have that the terms $a_n x_0^n \to 0$ as $n \to \infty$.

Being convergent, the sequence $(a_n x_0^n)$ is bounded: there is a real $M > 0$ such that we have $|a_n x_0^n| \leq M$ for all $n = 0, 1, 2, 3, \ldots$. For any $x \in \mathbb{R}$ that satisfies $|x| < |x_0|$, we have

$$|a_n x^n| = |a_n x_0|^n \left| \frac{x}{x_0} \right|^n \leq M \left| \frac{x}{x_0} \right|^n.$$

Since $|x/x_0| < 1$, we have that the geometric series

$$\sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n$$

converges, and so by the Algebraic Limit Theorem for Series (Theorem 2.7.1) we have that

$$\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$$

converges as well.

By the Comparison Theorem (Theorem 2.7.4) we have that

$$\sum_{n=0}^{\infty} |a_n x^n|$$

converges, and so by the Absolute Convergence Test (Theorem 2.7.6) we have that

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely as well. □

Lemma 6.5.3 (Abel's Lemma). Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$. Let $\sum_{n=1}^{\infty} a_n$ be a series whose sequence of partial sums $s_m = \sum_{n=1}^{m} a_n$ is bounded: there is $A > 0$ such that $|s_m| \leq A$ for all $m \in \mathbb{N}$. Then for integers $n > m \geq 0$ there holds

$$\left| \sum_{j=m+1}^{n} a_j b_j \right| \leq 2Ab_{m+1}.$$

Proof. Each s_m is defined for $m \geq 1$. We will need an s_0 which we set to 0.

We first show the "summation by parts" formula

$$\sum_{j=m+1}^{n} a_j b_j = s_n b_{n+1} - s_m b_{m+1} + \sum_{j=m+1}^{n} s_j (b_j - b_{j+1})$$

holds for all $n \geq 1$ and all $m \geq 0$. [Think of this like “integration by parts.”]

Starting with the right hand side of this formula, we have for $m \geq 1$ that

$$
\begin{aligned}
&s_n b_{n+1} - s_m b_{m+1} + \sum_{j=m+1}^{n} s_j (b_j - b_{j+1}) \\
&= (a_1 + \cdots + a_n) b_{n+1} - (a_1 + \cdots + a_m) b_{m+1} \\
&\quad + (a_1 + \cdots + a_m + a_{m+1}) b_{m+1} - (a_1 + \cdots + a_m + a_{m+1}) b_{m+2} \\
&\quad + (a_1 + \cdots + a_{m+1} + a_{m+2}) b_{m+2} - (a_1 + \cdots + a_{m+1} + a_{m+2}) b_{m+3} \\
&\quad \vdots \\
&\quad + (a_1 + \cdots + a_{n-2} + a_{n-1}) b_{n-1} - (a_1 + \cdots + a_{n-2} + a_{n-1}) b_n \\
&\quad + (a_1 + \cdots + a_{n-1} + a_n) b_n - (a_1 + \cdots + a_n) b_{n+1} \\
&= a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + \cdots + a_n b_n \\
&= \sum_{j=m+1}^{n} a_j b_j,
\end{aligned}
$$

which is the left hand side of the formula.

The only difference in the case of $m = 0$ are the second and third terms after the first equal sign above, which result in a term of $a_1 b_1$.

By this “summation by parts” formula, for $m \geq 0$, we have

$$
\begin{aligned}
\left| \sum_{j=m+1}^{n} a_j b_j \right| &= \left| s_n b_{n+1} - s_m b_{m+1} + \sum_{j=m+1}^{n} s_j (b_j - b_{j+1}) \right| \\
&\leq |s_n| b_{n+1} + |s_m| b_{m+1} + \sum_{j=m+1}^{n} |s_j| (b_j - b_{j+1}) \\
&= |s_n| b_{n+1} + |s_m| b_{m+1} + |s_m| (b_{m+1} - b_{m+2}) + \cdots + |s_n| (b_n - b_{n+1}) \\
&\leq A b_{n+1} + A b_{m+1} + A (b_{m+1} - b_{m+2}) + \cdots + A (b_n - b_{n+1}) \\
&= A b_{m+1} + A b_{m+1} \\
&= 2 A b_{m+1},
\end{aligned}
$$

which establishes the desired inequality. □

Proof of Abel’s Theorem. Applying the Cauchy Criterion to the convergent series $\sum_{n=0}^{\infty} a_n R^n$ we have for each $\epsilon > 0$ the existence of $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have

$$
|a_{m+1} R^{m+1} + a_{m+2} R^{m+2} + \cdots + a_n R^n| < \frac{\epsilon}{3}.
$$

We are driving for the Cauchy Criterion for Uniform Convergence on the interval $[0, R]$.

To this end, we have for any $x \in [0, R]$ and $n > m \geq N$ that

$$
\begin{aligned}
&a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_n x^n \\
&= a_{m+1} R^{m+1} \left(\frac{x}{R} \right)^{m+1} + a_{m+2} R^{m+2} \left(\frac{x}{R} \right)^{m+2} + \cdots + a_n R^n \left(\frac{x}{R} \right)^n .
\end{aligned}
$$

We then apply Lemma 6.5.3 to the sequences $(a_n R^n)$ and $(x/R)^n$ to get for all $n > m \geq N$ and all $x \in [0, R]$ that

$$\begin{aligned}&\left| a_{m+1} R^{m+1} \left(\frac{x}{R}\right)^{m+1} + a_{m+2} R^{m+2} \left(\frac{x}{R}\right)^{m+2} + \cdots + a_n R^n \left(\frac{x}{R}\right)^n \right| \\ &\leq 2 \left(\frac{\epsilon}{3}\right) \left(\frac{x}{R}\right)^{m+1} \\ &< \epsilon.\end{aligned}$$

By the Cauchy Criterion for Uniform Convergence on the interval $[0, R]$ (Theorem 6.2.5), we have the uniform convergence of the power series on $[0, R]$. $\square$

Proof of Theorem 6.5.6. We work towards the Weierstrass M-Test to get uniform convergence of the term by term differentiated power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ on compact subintervals of $(-R, R)$.

To this end, let $x_1 \in (0, R)$ and pick t satisfying $x_1 < t < R$.

Then for all x satisfying $|x| \leq x_1$, we have

$$|n a_n x^{n-1}| = \frac{1}{t} \left(n \left| \frac{x^{n-1}}{t^{n-1}} \right| \right) |a_n t^n| \leq \frac{1}{t} \left(n \left| \frac{x_1^{n-1}}{t^{n-1}} \right| \right) |a_n t^n|.$$

Since $\sum_{n=0}^{\infty} a_n t^n$ converges we have that the terms $a_n t^n$ are bounded: there is $L > 0$ such that $|a_n t^n| \leq L$ for all $n = 0, 1, 2, 3, \ldots$, and so

$$|n a_n x^{n-1}| \leq \frac{L}{t} \left(n \left| \frac{x_1^{n-1}}{t^{n-1}} \right| \right) = M_n$$

for all $|x| \leq x_1$ and all $n \in \mathbb{N}$.

For $s = |x_1/t|$ and $b_n = n s^{n-1}$ we have that $M_n = (L/t) b_n$ where

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{(n+1)s^n}{n s^{n-1}} = \lim_{n \to \infty} \frac{(n+1)s}{n} = s.$$

Since $0 < s < 1$, there exists $N \in \{0\} \cup \mathbb{N}$ such that for all $n \geq N$ we have that

$$\frac{b_{n+1}}{b_n} < 1.$$

Pick $0 < r < 1$ that for all $n \geq N$ satisfies

$$\frac{b_{n+1}}{b_n} < r.$$

Then we have that $b_{N+1} < b_N r$, and by induction that $b_{N+k} < b_N r^k$ for all $k \geq 1$.

The geometric series

$$\sum_{k=1}^{\infty} b_N r^k$$

converges because $0 < r < 1$, and so by the Comparison Test, the series

$$\sum_{n=N+1}^{\infty} b_n = \sum_{n=N+1}^{\infty} ns^{n-1}$$

converges as well.

Thus the series

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{L}{t}\left(n\left|\frac{x^{n-1}}{t^{n-1}}\right|\right)$$

converges, so that by the Weierstrass M-Test, the series

$$\sum_{n=1}^{\infty} na_n x^{n-1}$$

converges uniformly on the interval $[-x_1, x_1]$, and hence pointwise on $[-x_1, x_1]$.

Since x_1 satisfying $0 < x_1 < R$ is arbitrary, we have convergence of the term by term differentiated series $\sum_{n=1}^{\infty} na_n x^{n-1}$ at every point x in $(-R, R)$. $\square$