Math 341 Lecture #32
§6.6: Taylor Series

Most of the functions you have seen in Calculus, such as arctan, exp, and ln, can be written, or have a representation, as a convergent power series.

What about other familiar functions in Calculus?

Example 6.6.1. Find a power series representation for arctan.

We know that

$$\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2},$$

and that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots$$

for $t \in (-1, 1)$ (the convergent geometric series).

Replacing t with $-t^2$ in the geometric series give

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots$$

which is a convergent power series on the interval $(-1, 1)$, i.e., the radius of convergence is $R = 1$.

By the Fundamental Theorem of Calculus we know that

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt.$$

Now assuming that we can integrate a convergent power series term by term to get another convergent power series (on the same open interval of convergence) we have that

$$\int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \{1 - t^2 + t^4 - t^6 + \cdots\}\, dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

and so we get the power series representation

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \text{ for } |x| < 1.$$

You will investigate the convergence at $x = 1$ (homework problem 6.6.1).

We will rigorous justify that the integration of convergent power series term by term gives a convergent power series in Chapter 7.

Now we turn to the question of the relationship between a convergent power series $f(x)$ and the coefficients a_n in it.

Theorem 6.6.2 (Taylor's Formula). If, for some $R > 0$, we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,\ |x| < R,$$

then

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. To get the value of a_0 we set $x = 0$ in the power series to get

$$f(0) = \sum_{n=0}^{\infty} a_n 0^n = a_0.$$

To get the value of a_1 we set $x = 0$ in the first derivative to get

$$f'(0) = \sum_{n=1}^{\infty} n a_n 0^{n-1} = 1a_1.$$

To get the value of a_2 we set $x = 0$ in the second derivative to get

$$f''(0) = \sum_{n=2}^{\infty} n(n-1) a_n 0^{n-2} = (2)(1)a_2 = 2!a_2.$$

Continuing this pattern we get $f^{(n)}(0) = n!a_n$, so that

$$a_n = \frac{f^{(n)}(0)}{n!}$$

for each $n = 0, 1, 2, 3, \ldots$. □

The converse of Taylor's Formula asks if we have an infinitely differentiable function $f(x)$ defined on an open interval I centered at 0, does the Taylor series for f,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

converge to $f(x)$ on I?

Example. The function $f(x) = \sin(x)$ is infinitely differentiable on $\mathbb{R}$. Since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(4)}(0) = -1$, with the pattern repeating forever, the Taylor series for sin is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Does this Taylor's series converge to $\sin(x)$?

For a Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

convergence is in terms of the sequence of partial sums

$$s_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n = a_0 + a_1 x + \cdots + a_N x^N.$$

We are asking whether for each x in the interval of convergence do we have

$$\lim_{N\to\infty} s_N(x) = f(x).$$

Recast we are asking if the error function

$$E_N(x) = f(x) - s_N(x)$$

goes to 0 as $n \to \infty$.

Theorem 6.6.1 (Lagrange's Theorem). Let f be infinitely differentiable on $(-R, R)$ for $R > 0$, and define $a_n = f^{(n)}(0)/n!$. For a nonzero $x \in (-R, R)$, there exists a point c satisfying $|c| < |x|$ such that

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}.$$

See the Appendix for a proof.

Example. The Taylor series of $f(x) = \cos(x)$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

For every $x \neq 0$, there is by Lagrange's Theorem a c satisfying $|c| < |x|$ such that

$$|\cos(x) - s_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right|.$$

Since $|f^{(N+1)}(x)|$ is either $|\cos(x)|$ or $|\sin x|$, we have that $|f^{(N+1)}(c)| \leq 1$.
Restricting x to the compact interval $[-K, K]$ we then have that

$$|\cos(x) - s_N(x)| \leq \frac{K^{N+1}}{(N+1)!} \to 0.$$

Since K is arbitrary, we conclude that for every $x \in \mathbb{R}$ we have that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Is there an infinitely differentiable function $f(x)$ whose Taylor series does not converge to $f(x)$?

Counterexample. Consider the function

$$g(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

This function is infinitely differentiable when $x \neq 0$.

Is g differentiable at 0? Well,

$$\lim_{x\to 0^-} \frac{g(x)-g(0)}{x-0} = 0,$$

while

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+} \frac{\exp(-1/x^2)}{x} = \lim_{x\to 0^+} \frac{1/x}{\exp(1/x^2)}.$$

This is a ∞/∞ situation, so by L'Hospital's Rule we have

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+} \frac{-1/x^2}{(-2/x^3)\exp(1/x^2)} = \lim_{x\to 0^+} \frac{x}{2\exp(1/x^2)} = 0.$$

We we have that g is differentiable at 0 with $g'(0) = 0$.

Similarly, we can show that g is infinitely differentiable at 0 with $g^{(n)}(0) = 0$.

Thus the Taylor series for $g(x)$ is

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = 0 + 0x + 0x^2 + 0x^3 + \cdots = 0.$$

For $x > 0$ this Taylor series does not converge to $g(x)$ since $g(x) > 0$ while the Taylor series is 0.

Appendix. Proof of Lagrange's Remainder Theorem. The Taylor coefficients $a_0, a_1, \ldots, a_N$ have been chosen so that f and s_N have the same derivatives at $x = 0$, i.e.,

$$f^{(n)}(0) = s_N^{(n)}(0),\ n = 0, 1, 2, \ldots, N,$$

with $s_N^{(N+1)} = 0$. Thus the error function satisfies

$$E_N^{(n)}(0) = 0,\ n = 0, 1, 2, \ldots, N.$$

WLOG suppose $x > 0$. By the Generalized Mean Value Theorem to the functions E_N and x^{N+1} on the interval $[0, x]$: there exists $x_1 \in (0, x)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N}.$$

We then apply the Generalized Mean Value Theorem to the functions $E_N'(x)$ and $(N+1)x^N$ on the interval $[0, x_1]$: there exists $x_2 \in (0, x_1)$ such that

$$\frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)Nx_2^{N-1}}.$$

Continuing to apply the Generalized Mean Value Theorem we obtain $x_{N+1} \in (0, x_N) \subseteq \cdots \subseteq (0, x)$ satisfying

$$\frac{E_N^{(N)}(x_N)}{(N+1)N\cdots 2x_N} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}.$$

Thus the first and last terms are the same:

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}.$$

Since $s_N^{(N+1)}(x_{N+1}) = 0$, we have $E_N^{(N+1)}(x_{N+1}) = f^{(N+1)}(x_{N+1})$, so that with $c = x_{N+1}$, we obtain

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$