Math 341 Lecture #34
§7.3: Integrating Functions with Discontinuities

We have seen that a function continuous on a compact interval $[a, b]$ is integrable.

But what about a function not continuous on $[a, b]$?

Example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1, \\ 0 & \text{if } x = 1, \end{cases}$$

on the compact interval $[0, 2]$.

This bounded function has a discontinuity at $x = 1$, but is this enough to prevent integrability?

To show that f is integrable, we will use the Integrability Criterion (Theorem 7.2.8) by finding for each $\epsilon > 0$ a partition P_ϵ of $[0, 2]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

The way to choose P_ϵ is to reduce the contribution to $L(f, P_\epsilon)$ that the discontinuity presents.

Let $P_\epsilon = \{0, 1 - \epsilon/3, 1 + \epsilon/3, 2\}$.

Then

$$U(f, P_\epsilon) = 1(1 - \epsilon/3) + 1(2\epsilon/3) + 1(1 - \epsilon/3) = 2.$$

The subinterval of P that contains $x = 1$ is small, so that

$$L(f, P_\epsilon) = 1(1 - \epsilon/3) + 0(2\epsilon/3) + 1(1 - \epsilon/3) = 2 - \frac{2\epsilon}{3}.$$

Putting things together gives us

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \frac{2\epsilon}{3} < \epsilon$$

and hence f is integrable with

$$\int_0^1 f = 2.$$

The integrability of this nearly constant f is because we were able to isolate the single discontinuity of f within a small subinterval of the partition.

Using this isolation, we will show that any bounded function with a single discontinuity is integrable, first when the discontinuity occurs at an endpoint.

Theorem 7.3.2. Suppose $f : [a, b] \to \mathbb{R}$ is bounded. If for every $c \in (a, b)$, we have that f is integrable on $[c, b]$, then f is integrable on $[a, b]$. An analogous result holds at the other endpoint.

Proof. Let $M > 0$ satisfy $|f(x)| \leq M$ for all $x \in [a, b]$.

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be partition of $[a, b]$.

A partition of $[x_1, b]$ is obtained from P by deleting a from P:

$$P_{[x_1,b]} = \{x_1 < x_2 < \cdots < x_n = b\}.$$

To make use of Theorem 7.2.8 to get integrability of f on $[a, b]$, we consider

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= (M_1 - m_1)(x_1 - a) + \sum_{k=2}^{n} (M_k - m_k)\Delta x_k \\ &= (M_1 - m_1)(x_1 - a) + U(f, P_{[x_1,b]}) - L(f, P_{[x_1,b]}), \end{aligned}$$

which we want to make smaller than a given $\epsilon > 0$.

How do we control the term $(M_1 - m_1)(x_1 - a)$?

Well, since $|f(x)| \leq M$ for all $x \in [a, b]$, we have $-M \leq f(x) \leq M$, so that $-M \leq m_1$ and $M_1 \leq M$, and hence

$$M_1 - m_1 \leq M + M = 2M.$$

Choosing x_1 so that

$$x_1 - a \leq \frac{\epsilon}{4M}$$

we have that

$$(M_1 - m_1)(x_1 - a) \leq 2M\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

How do we control the term $U(f, P_{[x_1,b]}) - L(f, P_{[x_1,b]})$?

Since f is integrable on $[x_1, b]$, there is a partition P_1 of $[x_1, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}.$$

For the partition $P_2 = \{a\} \cup P_1$ of $[a, b]$, and using P_2 in place of P and P_1 in place of $P_{[x_1,b]}$, we then have

$$\begin{aligned} U(f,P_2) - L(f,P_2) &\leq 2M(x_1 - a) + U(f,P_1) - L(f,P_1) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

By Theorem 7.2,8, the function f is integrable on $[a, b]$. $\square$

To get integrability of a bounded function f on $[a, b]$ with a single discontinuity at an interior point c, we use at result to be proved in the next Lecture, that if f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$.

By induction we can get the integrability of a bounded function f on $[a, b]$ which has a finite number of discontinuities.

This then points to the question: How large does the set of discontinuities have to be before a bounded function is not integrable?