Math 341 Lecture #35
§7.4: Properties of the Integral, Part I

We establish a result about the Riemann integral that we assumed in the previous lecture to show that a bounded function with finitely many discontinuities is integrable.

This will require manipulating partitions.

Theorem 7.4.1. Assume $f : [a,b] \to \mathbb{R}$ is bounded, and let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and on $[c,b]$. In this case we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Remark. We are suppressing the dx in the notation for the integral. It is understood that we are using dx to measure the lengths of the subintervals in a partition of $[a,b]$.

Proof. Suppose that f is integrable on $[a,b]$.

Then for each $\epsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$.

If $c \in P$, then we define partitions $P_1 = P \cap [a,c]$ of $[a,c]$ and $P_2 = P \cap [c,b]$ of $[c,b]$.

If $c \notin P$, then we refine P to include c, which refinement doesn't alter $U(f,P)-L(f,P) < \epsilon$, and proceed as above to form the partitions P_1 and P_2.

In the partition P we have $c = x_l$ for some $0 < l < n$, and so

$$\begin{aligned} \epsilon &> U(f,P) - L(f,P) \\ &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &\geq \sum_{k=1}^{l} (M_k - m_k)\Delta x_k \\ &= U(f,P_1) - L(f,P_2) \end{aligned}$$

because $M_k - m_k \geq 0$ for all $1 \leq k \leq n$.

Similarly we also get that

$$\epsilon > U(f,P) - L(f,P) \geq U(f,P_2) - L(f,P_2).$$

Thus f is integrable on $[a,c]$ and f is integrable on $[c,b]$.

Now suppose that f is integrable on $[a,c]$ and f is integrable on $[c,b]$.

For $\epsilon > 0$ there are partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$\begin{aligned} U(f,P_1) - L(f,P_1) &< \frac{\epsilon}{2}, \\ U(f,P_2) - L(f,P_2) &< \frac{\epsilon}{2}. \end{aligned}$$

In the partition $P = P_1 \cup P_2$ of $[a,b]$ there is $0 < l < n$ such that $x_l = c$, and so

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= \sum_{k=1}^{l} (M_k - m_k)\Delta x_k + \sum_{k=l+1}^{n} (M_k - m_k)\Delta x_k \\ &= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus f is integrable on $[a,c]$.

It remains to show that

$$\int_a^b f = \int_a^c f + \int_c^b f$$

for which we use the partitions P_1 and P_2 above.

With $P = P_1 \cup P_2$ we have that $U(f,P) - L(f,P) < \epsilon$, so that

$$U(f,P) < L(f,P) + \epsilon.$$

Because $P = P_1 \cup P_2$ where $x_l = c$, we have

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = \sum_{k=1}^{l} (M_k - m_k)\Delta x_k + \sum_{k=l+1}^{n} (M_k - m_k)\Delta x_k \\ &= L(f,P_1) + L(f,P_2). \end{aligned}$$

Thus we have

$$\begin{aligned} \int_a^b f \le U(f,P) &< L(f,P) + \epsilon \\ &= L(f,P_1) + L(f,P_2) + \epsilon \\ &\le \int_a^c f + \int_c^b f + \epsilon, \end{aligned}$$

which, since $\epsilon > 0$ is arbitrary, implies that

$$\int_a^b f \le \int_a^c f + \int_c^b f.$$

To get the other inequality, we recall that for the partitions P_1 and P_2 we have

$$\begin{aligned} U(f,P_1) - L(f,P_1) &< \frac{\epsilon}{2}, \\ U(f,P_2) - L(f,P_2) &< \frac{\epsilon}{2}, \end{aligned}$$

so that

$$\begin{aligned}\int_a^c f + \int_c^b f &\leq U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + L(f, P_2) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= L(f, P) + \epsilon \\ &\leq \int_a^b f + \epsilon.\end{aligned}$$

Again, the arbitrariness of $\epsilon > 0$ implies that

$$\int_a^c f + \int_c^b f \leq \int_a^b f.$$

Thus we obtain

$$\int_a^b f = \int_a^c f + \int_c^b f$$

as believed. □

Rather than dealing with partitions all the time, we recall some familiar properties of the integral.

Theorem 7.4.2. Assume that f and g are integrable functions on $[a, b]$.

(i) The function $f + g$ is integrable on $[a, b]$ with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
(ii) For $k \in \mathbb{R}$, the function kf is integrable with $\int_a^b kf = k \int_a^b f$.
(iii) If $m \leq f \leq M$, then $m(b - a) \leq \int_a^b f \leq M(a - b)$.
(iv) If $f \leq g$ then $\int_a^b f \leq \int_a^b g$.
(v) The function $|f|$ is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. (i) This follows by showing that $U(f+g, P) \leq U(f, P)+U(g, P)$ and $L(f+g, P) \geq L(f, P) + L(g, P)$ for all partitions P.

For part (ii) we use the integrability of f to find partitions P_n of $[a, b]$ such that $U(f, P_n)-L(f, P_n) < 1/n$.

Then for $k \geq 0$ we show that $U(kf, P_n) = kU(f, P_n)$ and $L(kf, P_n) = kL(f, P_n)$ so that $U(kf, P_n) - L(kf_n, P_n) < k/n$ which goes to 0.

Thus kf is integrable on $[a, b]$, and $\int_a^b kf = k \int_a^b f$.

What happens when $k < 0$?

Part (iii) follows by taking the trivial partition $P = \{a, b\}$.

Part (iv) follows by setting $h = g - f \geq 0$ and applying parts (i), (ii), and (iii).

Parti (v) is a Homework Problem (Exercises 7.4.1). □