Math 113 Lecture #1
Review of the FTC

The Fundamental Theorem of Calculus (FTC), Part I. If f is continuous on $[a, b]$, then the function

$$g(x) = \int_a^x f(t)\,dt, \text{ for } t \text{ in } [a, b],$$

is continuous on $[a, b]$, is differentiable on (a, b), and $g'(x) = f(x)$.

The function g so constructed is called an *antiderivative* of f, i.e., an antiderivative of a continuous f is a differentiable F such that $F' = f$.

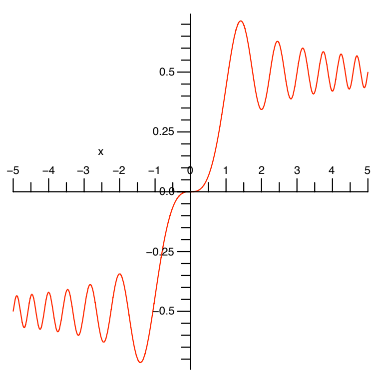
This part of the FTC tells us that **every** function continuous on a closed interval $[a, b]$ has an antiderivatve.

Example. Many functions useful in science and engineering are defined by Part I of the FTC, such as the Fresnel function

$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

used in the theory of diffraction of light waves, and more recently in highway design.

Here is the graph of the Fresnel function.



Since the integrand $f(t) = \sin(\pi t^2/2)$ is continuous on any closed interval, the derivative of the Fresnel function is $S'(x) = \sin(\pi x^2/2)$.

What is the second derivative of $S(x)$?

The function $S'(x) = \sin(\pi x^2/2)$ is a composition of differentiable functions, and so is differentiable.

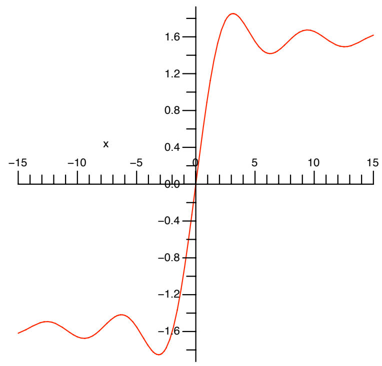
Then, by the chain rule, we have

$$S''(x) = \pi x \cos(\pi x^2/2).$$

Example. Another function defined by integration is the sine integral function,

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

which is used in electrical engineering. Here is the graph of the sine integral function.



Is the integrand $f(t) = \dfrac{\sin t}{t}$ continuous on any closed interval? Yes, because

$$\lim_{t \to 0} \frac{\sin t}{t} = 1.$$

The derivative of the sine integral function is $\mathrm{Si}'(x) = \dfrac{\sin x}{x}$, which equals 1 when $x = 0$.

Is the sine integral function twice differentiable?

Yes, it is, but you can not apply the quotient rule of differentiation to get $\mathrm{Si}''(x)$.

FTC, Part II. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is any antiderivative of f.

It is this part of the FTC that provides a way to evaluate exactly the value of definite integrals.

Although Part I of the FTC states that every continuous f has an antiderivative F, the tricky part is sometimes in finding F.

Example. An antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, i.e., $F'(x) = f(x)$. Thus

$$\int_{-\pi/2}^{\pi/2} \cos t\, dt = F(\pi/2) - F(-\pi/2) = \sin(\pi/2) - \sin(-\pi/2) = 1 + 1 = 2.$$