Math 113 Lecture #2Review of the Substitution Rule

The Substitution Rule for Indefinite Integrals. If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$

Example. Evaluate the indefinite integral $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$.

Make the substitution $u = 1 + \sin x$.

Then $du = \cos x \, dx$ (i.e. $u' = \cos x$), from which follows

$$\int \frac{\cos x}{\sqrt{1+\sin x}} \, dx = \int \frac{1}{\sqrt{u}} \, du = \int u^{-1/2} \, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} = 2\sqrt{1+\sin x} + C.$$

We can use differentiation to verify this calculation:

$$\frac{d}{dx}2\sqrt{1+\sin x} = 2\frac{d}{dx}\sqrt{1+\sin x} = 2\left(\frac{1}{2}\right)(1+\sin x)^{-1/2}\cos x = \frac{\cos x}{\sqrt{1+\sin x}}$$

Example. Evaluate

$$\int x^5 \sqrt{1+x^2} \, dx.$$

Make the substitution $u = 1 + x^2$.

Then

$$du = 2xdx$$
, or $(1/2)du = xdx$.

We can borrow an x from x^5 , but what do we do with the extra x^4 ? Since $u = 1 + x^2$, we have $x^4 = (u - 1)^2$, so by the substitution rule we obtain

$$\int x^5 \sqrt{1+x^2} \, dx = \frac{1}{2} \int (u-1)^2 \sqrt{u} \, du$$

The new integral simplifies and integrates to

$$\frac{1}{2} \int \left(u^{5/2} - 2u^{3/2} + u^{1/2} \right) du = \frac{1}{2} \left(\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} \right) + C$$
$$= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C.$$

Returning to the original independent variable we have

$$\int x^5 \sqrt{1+x^2} \, dx = \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C.$$

By differentiation and a bit of algebra we can verify that we computed the indefinite integral correctly:

$$\begin{aligned} \frac{d}{dx} \left(\frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} \right) \\ &= x(1+x^2)^{5/2} - 2x(1+x^2)^{3/2} + x(1+x^2)^{1/2} \\ &= x(1+x^2)^{1/2} \left((1+x^2)^2 - 2(1+x^2) + 1 \right) \\ &= x(1+x^2)^{1/2} \left(1 + 2x^2 + x^4 - 2 - 2x^2 + 1 \right) \\ &= x^5 \sqrt{1+x^2}. \end{aligned}$$

The Substitution Rule for Definite Integrals. If g' is continuous on [a, b] and f is continuous on the range of u = g(x), then

$$\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

Example. Evaluate $\int_0^2 v^2 \cos(v^3) dv$. Make the substitution $u = v^3$.

Then

$$du = 3v^2 dv$$
, or $(1/3)du = v^2 dv$,

and

$$u(0) = 0^3 = 0$$
 and $u(2) = 2^3 = 8$.

 So

$$\int_0^2 v^2 \cos(v^3) \, dv = \frac{1}{3} \int_0^8 \cos u \, du = \frac{1}{3} \left[\sin u \right]_{u=0}^{u=8} = \frac{\sin(8) - \sin(0)}{3} = \frac{\sin(8)}{3}$$

Why did we not go back to the original independent variable here?

We could have gone back to the original independent variable to get

$$\int_0^2 v^2 \cos(v^3) \, dv = \left[\frac{\sin(v^3)}{3}\right]_{v=0}^{v=2} = \frac{\sin(2^3) - \sin(0)}{3} = \frac{\sin(8)}{3}$$

How would we verify either calculation?

In the first, we can check that the substitution $u = v^3$ is carried through correctly, and then that the antiderivative of $\cos u$ is correct.

In the second (longer) calculation, we can verify that we have a correct antiderivative:

$$\frac{d}{dv}\frac{\sin(v^3)}{3} = \frac{3v^2\cos(v^3)}{3} = v^2\cos(v^3).$$

Example. Evaluate $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx.$

The choice of a substitution is not clear. After a few trials and errors, one might try the substitution $u = x^2$.

For this choice,

$$du = 2xdx$$
 or $\frac{1}{2}du = xdx$

and

$$u(-\pi/2) = \frac{\pi^2}{4}$$
 and $u(\pi/2) = \frac{\pi^2}{4}$

Then

$$\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} \, dx = \int_{\pi^2/4}^{\pi^2/4} \frac{u \sin(\sqrt{u})}{1+u^3} \, du = 0.$$

We can verify this by noticing that the integrand is an odd function, and we are integrating it over an interval symmetric about the origin, which means that the integral has to be 0.

Word Problems. Integration is used to solve word problems.

Example. An oil storage tank ruptures at time t = 0 and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ litres per minute. How much oil leaks out during the first hour?

Integration of the rate of leakage via u-substitution gives the answer:

$$\int_{0}^{60} r(t) dt = \int_{0}^{60} 100e^{-0.01t} dt \quad [u = -0.01t, \ du = -0.01dt]$$
$$= \int_{0}^{-0.6} (-10000)e^{u} du$$
$$= (-10000) \left[e^{u}\right]_{u=0}^{u=-0.6}$$
$$= (-10000)[e^{-0.6} - 1]$$
$$= 10000(1 - e^{-0.6})$$
$$\approx 4511.88.$$

So approximately 4512 litres of oil leaked out of the tank in 60 minutes.

How could you check this answer?

Because r(0) = 100 and r(t) is a decreasing function of t, the worst rate of leakage is 100 litres per minute. Over 60 minutes, that would mean no more than 6000 litres would leak out. Since r(t) decrease with t, we expect less than 6000 litres to leak out. So the answer of roughly 4512 litres is in the ball park.