Math 113 Lecture #3 §6.1: Areas Between Curves

Compute the Area Enclosed by Given Curves. A Riemann sum will approximate the area between the graphs of two functions, and the limit will give the exact area as that of a definite integral.

First Type. For $f(x) \ge g(x)$ on [a, b], the area above y = g(x) and below y = f(x) on [a, b] is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x = \int_a^b \left(f(x) - g(x) \right) \, dx.$$

Example. Compute the area enclosed by y = x + 1, $y = 9 - x^2$, x = -1, and x = 2. Here is a graph of these four curves.



There are actually three areas enclosed by these curves, but only one enclosed by all four. For the area enclosed by all four, the top curve is $y = f(x) = 9 - x^2$ on [-1, 2], and the bottom curve is y = g(x) = x + 1 on [-1, 2].

This area is the area between the graphs of f(x) and g(x) over the interval [-1, 2].

The area is

$$A = \int_{a}^{b} \left(f(x) - g(x) \right) \, dx = \int_{-1}^{2} \left(9 - x^2 - (x+1) \right) \, dx = \int_{-1}^{2} \left(9 - x^2 - x - 1 \right) \, dx$$
$$= \int_{-1}^{2} \left(8 - x - x^2 \right) \, dx = \left[8x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2} = 16 - \frac{4}{2} - \frac{8}{3} - \left(-8 - \frac{1}{2} - \frac{-1}{3} \right)$$
$$= 16 - 2 - \frac{8}{3} + 8 + \frac{1}{2} - \frac{1}{3} = 19 + \frac{1}{2} = \frac{39}{2}.$$

Second Type. For $f(y) \ge g(y)$ on [c, d], the area to the right of x = g(y) and to the left of x = f(y) is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(f(y_i^*) - g(y_i^*) \right) \Delta y.$$

The approximating rectangles are horizontal with "base" Δy and "height" $f(y_i^*) - g(y_i^*)$.

Example. Find the area enclosed by $4x + y^2 = 12$ and x = y. Here is a graph of the two curves.



For this area, the top curve is piece-wise defined, i.e., part of one curve, then part of another curve.

The endpoints of the left and right side curves are where the two curves intersect:

$$4x + y^{2} = 12 \text{ and } x = y \implies 4x + x^{2} = 12 \implies x^{2} + 4x - 12 = 0$$
$$\implies x = \frac{-4 \pm \sqrt{16 + 48}}{2} = \frac{-4 \pm 8}{2} = 2, -6,$$
$$\implies (-6, -6) \text{ and } (2, 2).$$

The left side is the graph of x = g(y) = y on [-6, 2], and the right side is the graph of $x = f(y) = (12 - y^2)/4$ on [-6, 2].

The area is between the graphs of g(y) and f(y) on [-6, 2].

The area is

$$\begin{split} A &= \int_{c}^{d} \left(f(y) - g(y) \right) \, dy = \int_{-6}^{2} \left(\frac{12 - y^{2}}{4} - y \right) \, dy = \int_{-6}^{2} \left(3 - y - \frac{y^{2}}{4} \right) \, dy \\ &= \left[3y - \frac{y^{2}}{2} - \frac{y^{3}}{12} \right]_{-6}^{2} = 6 - \frac{4}{2} - \frac{8}{12} - \left(-18 - \frac{36}{2} - \frac{-216}{12} \right) \\ &= 6 - 2 - \frac{2}{3} + 18 + 18 - 18 = 22 - \frac{2}{3} = \frac{64}{3}. \end{split}$$

More General Areas Between Curves. When the area is enclosed by y = g(x) and y = f(x) on [a, b], but $f(x) \ge g(x)$ does not hold on [a, b], then the area is

$$\int_{a}^{b} |f(x) - g(x)| \, dx.$$

The absolute value accounts for changes in the sign of f(x) - g(x), and thus ensures that the definite integral captures area, not the difference of signed areas.

One has to find the points of intersection of f and g, and split the integral into subintervals determined by those points of intersection, intervals on which either $f(x) \ge g(x)$ holds or $f(x) \le g(x)$ holds.

On a subinterval where $f(x) \ge g(x)$ the absolute value returns f(x) - g(x).

On a subinterval where $f(x) \leq g(x)$, the absolute value returns

$$-(f(x) - g(x)) = g(x) - f(x).$$

Similar statements holds for areas between x = g(y) and x = f(y) on [c, d].

Example. Find the area enclosed by $y = \cos x$ and $y = 1 - \cos x$ on $[0, \pi]$.



The two curves intersect:

$$\cos x = 1 - \cos x \implies \cos x = \frac{1}{2} \implies x = \frac{\pi}{3}.$$

The area enclosed is

$$A = \int_0^{\pi/3} (2\cos x - 1) \, dx - \int_{\pi/3}^{\pi} (2\cos x - 1) \, dx$$

= $\left[2\sin x - x \right]_0^{\pi/3} - \left[2\sin x - x \right]_{\pi/3}^{\pi}$
= $2\sin(\pi/3) - \frac{\pi}{3} - \left(0 - 0 \right) - \left[0 - \pi - \left(2\sin(\pi/3) - \frac{\pi}{3} \right) \right]$
= $4\sin(\pi/3) + \pi - \frac{2\pi}{3} = 4\frac{\sqrt{3}}{2} + \frac{\pi}{3} = 2\sqrt{3} + \frac{\pi}{3}.$