Math 113 Lecture #4
§6.2: Volumes

Defining Volume by Slicing. Slice a solid S lying along the x-axis, by n parallel planes $P_{x_1}, \ldots, P_{x_n}$ perpendicular to the x-axis and located at equally spaced points $x_1, \ldots, x_n$.

For each slab S_i so obtained, we choose a point x_i^* in $[x_{i-1}, x_i]$ at which to measure the cross-sectional area $A(x_i^*)$ of the part of S_i that lies in the plane $P_{x_i^*}$.

The volume of the i^{th} slab is

$$V(S_i) \approx A(x_i^*)\Delta x.$$

An approximation of the volume of the solid is the sum of the approximations of the volumes of the slabs of the slicing:

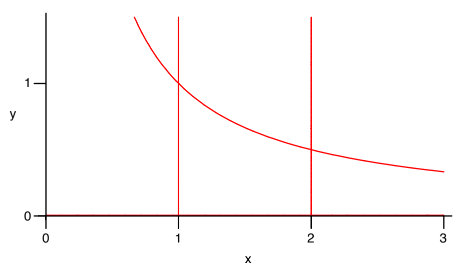
$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

If $A(x)$ be the cross-sectional area of S at x, then the volume of the solid S is

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx$$

provided the limit exists (which is the case if $A(x)$ is continuous on $[a, b]$).

Example 1. Let R be the region in the xy-plane enclosed by the curves $y = 1/x$, $x = 1$, $x = 2$, and $y = 0$. Here are these curves.



Revolving the region about the x-axis determines a solid S.

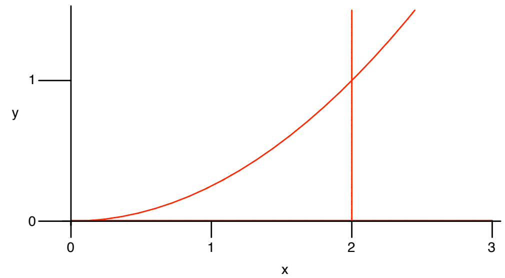
For this solid of revolution, each slab of the slicing is a disk with radius given by the top curve of the region:

$$A(x) = \pi\left(\frac{1}{x}\right)^2.$$

As this $A(x)$ is continuous on $[1, 2]$, the volume of the solid of revolution is

$$V = \int_1^2 A(x)\ dx = \int_1^2 \frac{\pi}{x^2}\ dx = \pi\left[\frac{-1}{x}\right]_1^2 = \pi\left(-\frac{1}{2} - (-1)\right) = \frac{\pi}{2}.$$

Example 2. Let R be the region in the xy-plane enclosed by the curves $y = x^2/4$, $x = 2$, and $y = 0$. Here are these curves.



Revolving the region R about the y-axis determines a solid S.

For this solid of revolution, each slab of the slicing is a washer with the outer radius given by the right curve, and the inner radius given by the left curve of the region.

The left curve is $y = x^2/4$, or solving it for x, it is $x = \sqrt{4y}$.

The radii needed are functions of y:

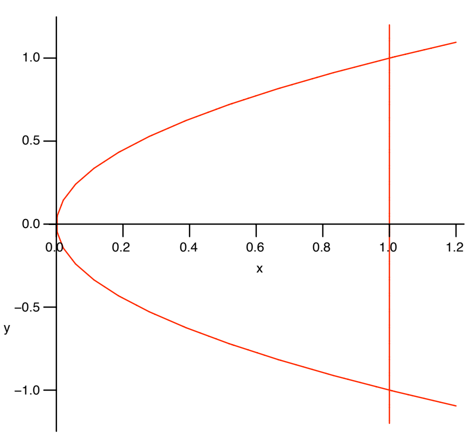
$$A(y) = \pi\big(2\big)^2 - \pi\big(\sqrt{4y}\big)^2 = 4\pi - 4\pi y = 4\pi(1-y).$$

As this $A(y)$ is continuous on $[0,1]$, the volume of the solid of revolution is

$$V = \int_0^1 A(y)\ dy = 4\pi \int_0^1 (1-y)\ dy = 4\pi\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi.$$

Example 3. Find the volume of the solid of revolution obtained by revolving about $x = 2$ the region enclosed by $x = y^2$ and $x = 1$.

Here is a graph of the region enclosed by the curves.



The curves intersect at $(1,1)$ and $(1,-1)$.

There is an outer radius of $2 - x$ and the inner radius of $2 - 1 = 1$.

The cross-sectional area is

$$A(y) = \pi(2-x)^2 - \pi(1)^2 = \pi\big(2 - y^2\big)^2 - \pi.$$

The volume of the new solid is

$$\begin{aligned}
V &= \int_{-1}^{1} A(y)\, dy = \pi \int_{-1}^{1} \left[(2-y^2)^2 - 1\right]\, dy \\
&= 2\pi \int_{0}^{1} \left[(2-y^2)^2 - 1\right]\, dy = 2\pi \int_{0}^{1} \left(4 - 4y^2 + y^4 - 1\right)\, dy \\
&= 2\pi \int_{0}^{1} \left(3 - 4y^2 + y^4\right)\, dy = 2\pi \left[3y - \frac{4y^3}{3} + \frac{y^5}{5}\right]_0^1 \\
&= 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56\pi}{15}.
\end{aligned}$$

Example 4. Find the volume of a pyramid of height h and base an equilateral triangle with side a where the horizontal slices are equilateral triangles (a solid known as a tetrahedron).

Draw this picture.

Label two vertices of the base by L and R, the center of the base O, and the top vertex of the solid T.

The distance between L and R is the given base length of a.

Let b be the length of the line from R to O.

Let the y-axis be situated so that the origin is at the center of the base O and the positive direction of y goes through the top vertex T.

At horizontal slice at height y is an equilateral triangle with two vertices A and B on the lines LT and RT respectively.

Let α be the side of this equilateral triangle, i.e., the length of the line AB.

Let M be center of the equilateral triangle at height y; the point M lies on the line OT.

Let β be the length of the line MB.

The two triangles LRO and ABM are similar, and so

$$\frac{a}{b} = \frac{\alpha}{\beta}.$$

This says that

$$\alpha = \frac{a\beta}{b}.$$

The triangles ORT and MBT are also similar, so that

$$\frac{b}{h} = \frac{\beta}{h-y}.$$

This says that

$$\beta = \frac{b(h-y)}{h}.$$

We can now determine the length of the side of the equilateral triangle at height y:

$$\alpha = \frac{a}{b} \cdot \beta = \frac{a}{b} \cdot \frac{b(h-y)}{h} = a\left(1 - \frac{y}{h}\right).$$

The area of the equilateral triangle at height y is thus

$$A(y) = \frac{1}{2} \cdot \alpha \cdot \frac{\sqrt{3}}{2}\alpha = \frac{\sqrt{3}}{4}a^2 \left(1 - \frac{y}{h}\right)^2$$

(where we have used the one-half base times height formula).

Finally, we can compute the volume of the tetrahedron:

$$\begin{aligned}
V &= \int_0^h A(y)\ dy \\
&= \int_0^h \frac{\sqrt{3}}{4}a^2 \left(1 - \frac{y}{h}\right)^2\ dy \\
&= \frac{\sqrt{3}\ a^2}{4} \int_0^h \left(1 - \frac{y}{h}\right)^2\ dy \quad [u = 1 - y/h, du = -(1/h)dy] \\
&= \frac{\sqrt{3}\ a^2}{4} \int_1^0 \left(-hu^2\right)\ du \\
&= \frac{\sqrt{3}\ a^2 h}{4} \int_0^1 u^2\ du \\
&= \frac{\sqrt{3}\ a^2 h}{4} \left[\frac{u^3}{3}\right]_0^1 \\
&= \frac{\sqrt{3}\ a^2 h}{12}.
\end{aligned}$$

Now if the height of the pyramid were equal to the side of the equilateral base of a, then the volume of the pyramid would be

$$V = \frac{\sqrt{3}\ a^3}{12}.$$