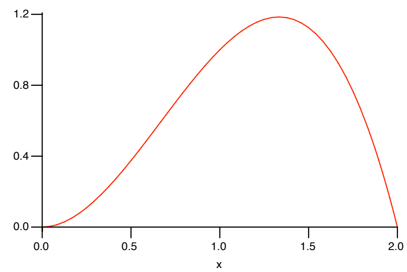
Math 113 Lecture #5
§6.3: Volume by Cylindrical Shells

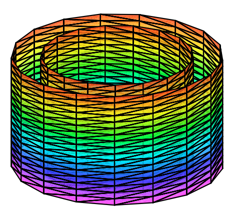
Definition of a Cylindrical Shell. Sometimes the method of disks (washers) is difficult to apply when computing the volume of a solid of revolution.

For instance, for the solid obtained by revolving the region



about the y-axis, the outer and inner radii needed require inverting the function on intervals where the function is monotone.

Instead, we consider the approximating rectangles in a Riemann sum for the area under the graph, and rotate these rectangles around the y-axis, which results in a cylindrical shell.



What is the volume of one of these cylindrical shells?

Say the outer cylindrical shell has radius r_2 and the inner has radius r_1.

Since the volume of a solid cylinder is $\pi(\text{radius})^2 \times$ height, the volume of the cylindrical shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} \, h(r_2 - r_1) \end{aligned}$$

Let $\Delta r = r_2 - r_1$, the thickness of the cylindrical shell, and let $r = (r_2 + r_1)/2$, the average of the outer and inner radii of the cylindrical shell.

The volume of the cylindrical shell is then

$$V = 2\pi r h \Delta r.$$

Here the factor $2\pi r$ is the average circumference of the cylindrical shell, the factor h is its height, and the factor Δr is its the thickness.

Computing Volumes of Solids of Revolution by Cylindrical Shells.

The volume of the i^{th} cylindrical shell in the approximation is

$$V_i \approx 2\pi \bar{x}_i f(\bar{x}_i)\Delta x.$$

The sum of the volumes of the n cylindrical shells that approximate the solid is

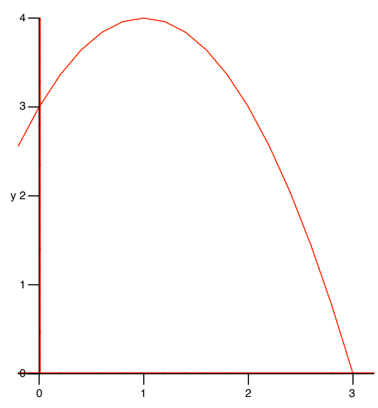
$$V = \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x.$$

Having assumed that f is continuous on $[a, b]$, we see that this Riemann sum has a limit as $n \to \infty$, and so a definite integral gives the volume of S:

$$V = \int_a^b 2\pi x f(x)\, dx.$$

Example 1. Let R be the region in the xy-plane enclosed by $y = 3 + 2x - x^2$, $y = 0$, and $x = 0$.

Here are the graphs of these curves, and the region R they enclose.



Let S be the solid obtained by revolving the region R about the y-axis.

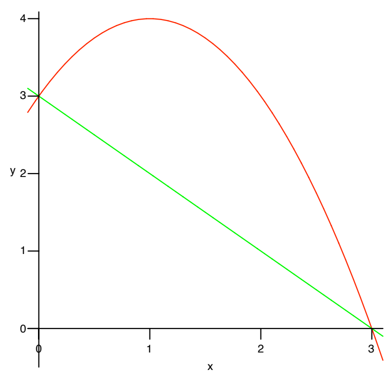
Can you visualize the cylindrical shells used to compute the volume?

[Sketch a cylindrical shell for this solid of revolution.]

Since the height is $f(x) = 3 + 2x - x^2$ over the interval $[0, 3]$, the volume of the solid by the method of cylindrical shells is

$$\begin{aligned} V &= \int_0^3 2\pi x f(x)\, dx \\ &= 2\pi \int_0^3 x(3 + 2x - x^2)\, dx \\ &= 2\pi \int_0^3 \left(3x + 2x^2 - x^3\right) dx \\ &= 2\pi \left[\frac{3x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4}\right]_0^3 \\ &= 2\pi \left[\frac{27}{2} + \frac{54}{3} - \frac{81}{4}\right] \\ &= \frac{45\pi}{2}. \end{aligned}$$

Example 2. Find the volume of the solid S obtained by revolving around the y-axis the region enclosed by $y = 3 + 2x - x^2$ and $x + y = 3$.



The height of a cylindrical shell for this solid is the difference of the top curve and the bottom curve.

Can you sketch a typical shell for this solid? The volume is

$$\begin{aligned} V &= \int_0^3 2\pi x\big(f(x) - g(x)\big)\, dx \\ &= 2\pi \int_0^3 x\big(3 + 2x - x^2 - (3 - x)\big)\, dx \\ &= 2\pi \int_0^3 x\big(3x - x^2\big)\, dx \\ &= 2\pi \int_0^3 (3x^2 - x^3)\, dx. \end{aligned}$$

The calculation is left for you to finish. [The answer is $27\pi/2$, if you wanted to know.]