

## Math 113 Lecture #6

### §6.4: Work

**Definition of Work.** The term work means total amount of effort required to perform a task, a quantity that depends on the force used in performing that task.

Newton's second law quantifies the force that is exerted on an object.

If an object is moved along a straight path, and  $s(t)$  is the position of the object along that line at time  $t$ , and  $m$  is the mass of the object, then the force exerted on that object is its mass times its acceleration:

$$F = m \frac{d^2 s}{dt^2}.$$

The units for force in the SI (i.e., metric) system are kilogram meters per second square ( $\text{kg m/s}^2$ ), or newtons (N), while in imperial measurement, the units of force are pounds (lb), which is the weight of the object.

When the acceleration is constant, the force  $F$  is constant, and the work done by this constant force while the object is displaced a distance of  $d$  units of length is defined to be

$$W = Fd.$$

The units of work in the SI system are newton-meters (N m), or joules (J), while in the imperial measurement system, the units of work are foot-pounds (ft-lb).

**Example 1.** What is the work done in lifting a 40 kg sandbag to a height of 1.5 metres?

Assuming that the sandbag is close enough to the earth's surface, the force exerted on the sandbag is equal and opposite to that exerted by gravity:

$$F = mg = 40 \times 9.8 = 392 \text{ N}.$$

So the work done in lifting the object 1.5 metres is

$$W = Fd = 392 \times 1.5 = 588 \text{ J}.$$

What happens if the force is not constant? How do we define and compute the work done?

Say the object moves along the  $x$ -axis in the positive direction from  $x = a$  to  $x = b$ , and that the force that acts on the object depends continuously on its position  $x$ .

Divide the interval  $[a, b]$  into subintervals of equal length  $\Delta x = (b - a)/n$ , where  $x_i = a + i\Delta x$  are the endpoints of the subintervals.

Choose sample points  $x_i^*$  in each subinterval  $[x_{i-1}, x_i]$ .

When  $n$  is large, the subintervals are small in length, and the values of the continuous  $f$  on  $[x_{i-1}, x_i]$  do not change much, so that  $f$  is nearly constant on that interval, and the value  $f(x_i^*)$  approximates that nearly constant function.

The work done in moving the object from  $x_{i-1}$  to  $x_i$  is therefore approximately

$$W_i = f(x_i^*)\Delta x.$$

An approximation of the work done in displacing the object from  $x = a$  to  $x = b$  is

$$W \approx \sum_{i=1}^n W_i = \sum_{i=1}^n f(x_i^*)\Delta x.$$

Since we assumed that  $f$  is continuous on  $[a, b]$ , the limit of the Riemann sum approximating the work done exists, is given by a definite integral, and is taken to define the work done in moving the object from  $x = a$  to  $x = b$ :

$$W = \int_a^b f(x) \, dx.$$

**Example 2.** When a particle is located a distance  $x$  metres from the origin, a force of  $\cos(\pi x/3)$  N acts on it.

How much work is done in moving the object from  $x = 1$  to  $x = 2$ ?

Since the force that acts on the object is continuous, the work done in moving the object from  $x = 1$  to  $x = 2$  is

$$\begin{aligned} W &= \int_a^b f(x) \, dx \\ &= \int_1^2 \cos\left(\frac{\pi x}{3}\right) \, dx \\ &= \left[ \frac{3}{\pi} \sin\left(\frac{\pi x}{3}\right) \right]_1^2 \\ &= \frac{3}{\pi} \left[ \sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \right] \\ &= 0. \end{aligned}$$

The object did move, so why is the work done 0?

From  $x = 1$  to  $x = 3/2$ , the force  $f(x) = \cos(\pi x/3)$  is positive, so that on this part of the displacement, the work done by the force is positive.

However, from  $x = 3/2$  to  $x = 2$ , the force is negative, so that on this part of the displacement, the work done is negative.

The magnitude of the work done on each segment of the displacement is the same, so that the work over the whole displacement sums to zero.

**Applications of Work.** Various laws or principles describe what the force is that acts on an object, and for displacements by such a force, the work done can be computed.

**Example 3.** Hooke's law states that the force required to keep a spring stretched  $x$  units from its natural length is proportional to  $x$ :

$$F(x) = kx,$$

where  $k$  is a positive constant (the so-called spring constant).

For a spring with a natural length of 20 cm, a force of 25 N is needed to keep the spring stretched to a length of 30 cm.

What is the work done in stretching the spring from 20 cm to 25 cm?

The force needed to keep the spring stretched at a length of  $30 - 20 = 10$  cm  $= 0.1$  m from its natural length is

$$25N = F(0.1) = 0.1k.$$

This implies that the spring constant is  $k = 250$  N/m.

The work done in stretching the spring from its natural length of 20 cm to 25 cm, a displacement from  $x = 0$  cm  $= 0$  m to  $x = 5$  cm  $= 0.05$  m is

$$W = \int_0^{0.05} 250x \, dx = 250 \left[ \frac{x^2}{2} \right]_0^{0.05} = 125(0.05)^2 = 0.3125 \text{ J}.$$

**Example 4.** A cable that weighs 2 lb/ft is used to lift 800 lbs of cargo and elevator up to the top of a 500 ft building. Find the work done.

The work done in moving the 800 lbs of cargo and elevator a distance of 500 ft is

$$W_{\text{cargo \& elevator}} = 800 \times 500 = 400,000 \text{ ft lbs}.$$

However, not every piece of the cable moves the same amount.

Place the  $x$ -axis along the cable, with  $x = 0$  at the top, and  $x = 500$  at the cargo and elevator.

Divide the interval  $[0, 500]$  into subintervals of equal lengths  $\Delta x = (500 - 0)/n$ , where the endpoints of the subintervals are  $x_i = i\Delta x$ , and choose points  $x_i^*$  in  $[x_{i-1}, x_i]$ .

The weight of the piece of cable lying in  $[x_{i-1}, x_i]$  is

$$F_i = \text{density} \times \text{length} = 2 \text{ lb/ft} \times \Delta x \text{ ft} = 2\Delta x \text{ lb}.$$

This piece of the cable moves approximately  $x_i^*$  feet, so that the work done for this piece is

$$W_i = (2\Delta x)(x_i^*) = 2x_i^* \Delta x.$$

A Riemann sum is emerging from this analysis, and so the work done in moving the cable is

$$W = \int_0^{500} 2x \, dx = \left[ x^2 \right]_0^{500} = 250,000.$$

The work done in moving the cable, cargo, and elevator is 650,000 ft-lbs.