

Math 113 Lecture #7

§6.5: Average Value of a Function

Definition of Average Value. The mean or average value of a finite set of numbers $y_1, \dots, y_n$ is the quantity

$$y_{\text{ave}} = \frac{y_1 + \dots + y_n}{n}.$$

One way to estimate the average daily temperature is to take hourly temperature measurements $y_1, \dots, y_{24}$, and average them.

However, this estimate of the average daily temperature uses very little of the daily temperature data available.

We like to think of the temperature as a continuous function of time, $T = f(t)$, and this is a reasonable thing to do.

How do we use more of the available daily temperature data in estimating an average daily temperature?

The best would be if we could use ALL of the daily temperature data to get the exact average daily temperature.

Mathematically, we are asking for the average value of a function f continuous on an interval $[a, b]$.

Dividing the interval $[a, b]$ into subintervals of equal length $\Delta = (b-a)/n$, with endpoints $x_i = a + i\Delta$, and choosing points x_i^* in $[x_{i-1}, x_i]$, we think of the average

$$\frac{f(x_1^*) + \dots + f(x_n^*)}{n}$$

as an estimate for all of the values of f on $[a, b]$.

How do we bridge the “gap” from a finite set of values of f to ALL the values of f ?

Since $\Delta x = (b-a)/n$, we can write $n = (b-a)/\Delta x$, and the average of the values of f at the points x_i^* becomes

$$\frac{f(x_1^*) + \dots + f(x_n^*)}{(b-a)/\Delta x} = \frac{f(x_1^*)\Delta x + \dots + f(x_n^*)\Delta x}{b-a} = \frac{1}{b-a} \sum_{i=1}^n f(x_i^*)\Delta x.$$

This estimate of the average of the values of f has a limit as $n \rightarrow \infty$ because we assumed that f is continuous.

We define the average value of f over $[a, b]$ by a definite integral:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

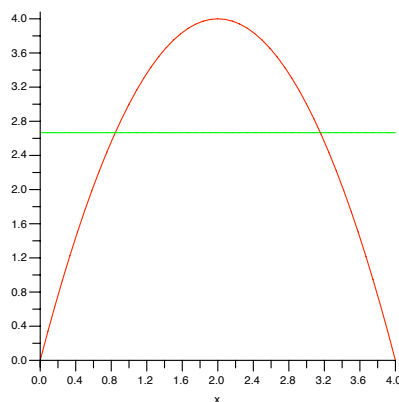
This definition gives the average value of an infinite set of data!

Computing the Average Value. Integration provides the means to compute the average value of function over an interval, i.e., the mean of an infinite set of data.

Example 1. What is the average value of $f(x) = 4x - x^2$ on $[0, 4]$?
It is

$$\begin{aligned}
 f_{\text{ave}} &= \frac{1}{b-a} \int_a^b f(x) \, dx \\
 &= \frac{1}{4-0} \int_0^4 (4x - x^2) \, dx \\
 &= \frac{1}{4} \left[2x^2 - \frac{x^3}{3} \right]_0^4 \\
 &= \frac{1}{4} \left[32 - \frac{64}{3} \right] \\
 &= \frac{1}{4} \left[\frac{32}{3} \right] \\
 &= \frac{8}{3}
 \end{aligned}$$

Here is the graph of f and its average value.



The Mean Value Theorem for Integrals. The average value formula can be written as

$$f_{\text{ave}}(b-a) = \int_a^b f(x) \, dx.$$

In other words, the definite integral of f over $[a, b]$ is the product of the average value of f over $[a, b]$ with the length of the interval $[a, b]$.

This says for a nonnegative function f that the area under the graph of f is the same as that of the rectangular box of base $b-a$ and height f_{ave} .

Is there a c in $[a, b]$ for which $f(c) = f_{\text{ave}}$?

In Example 1, there are actually two choices of c for which $f(c) = f_{\text{ave}}$.

Is this always the case?

Theorem. If f is continuous on $[a, b]$, then there is c in $[a, b]$ for which

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Proof. Recall the Mean Value Theorem for derivatives: if g is continuous on $[a, b]$ and differentiable on (a, b) , then there is c in (a, b) such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

Since f is continuous on $[a, b]$, the function

$$F(x) = \int_a^x f(t) \, dt$$

is continuous on $[a, b]$, differentiable on (a, b) , and $F'(x) = f(x)$ by the Fundamental Theorem of Calculus.

We now apply the Mean Value Theorem for derivatives to $F(x)$: there is c in (a, b) such that

$$F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} \left[\int_a^b f(x) \, dx - \int_a^a f(x) \, dx \right] = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

Since $F'(x) = f(x)$, we obtain

$$f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx = f_{\text{ave}},$$

which is what we wanted. □

Example 2. On a certain day, the temperature t hours past midnight is

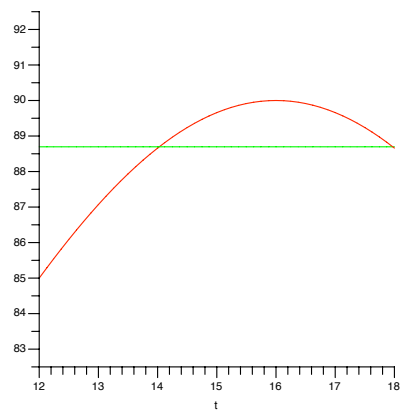
$$T(t) = 80 + 10 \sin \left(\frac{\pi(t - 10)}{12} \right).$$

What is the average temperature between noon and 6 p.m.? At what time between noon and 6 p.m. is this average temperature realized?

The average temperature sought is

$$\begin{aligned} \frac{1}{18 - 12} \int_{12}^{18} T(t) \, dt &= \frac{1}{6} \int_{12}^{18} \left[80 + 10 \sin \left(\frac{\pi(t - 10)}{12} \right) \right] dt \\ &= \frac{1}{6} \left[80t - \frac{120}{\pi} \cos \left(\frac{\pi(t - 10)}{12} \right) \right]_{12}^{18} \\ &= \frac{1}{6} \left[480 - \frac{120}{\pi} \left(\cos \left(\frac{2\pi}{3} \right) - \cos \left(\frac{\pi}{6} \right) \right) \right] \\ &= 80 - \frac{20}{\pi} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \right) \\ &= 80 + \frac{10 + 10\sqrt{3}}{\pi} \approx 88.7. \end{aligned}$$

Here are the graphs of the temperature and the average temperature over $[12, 18]$.



It appears the average temperature is realized twice between noon and 6 p.m.: at 2 p.m. and 6 p.m.