Math 113 Lecture #8§7.1: Integration by Parts

Integration by Parts for Indefinite Integrals. Just like the substitution rule for integration comes from the Chain Rule for differentiation, integration by parts is the integration rule that comes from the product rule for differentiation.

Recall for differentiable functions f(x) and g(x) that the product rule is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integration of this rule gives

$$f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx$$

Rearranging terms gives the rule for integration by part:

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.$$

Sometimes this rule is easier to remember when written differently: for

$$u = f(x)$$
 and $dv = g'(x)dx$,

then

$$du = f'(x)dx$$
 and $v = g(x)$,

so that integration by parts becomes

$$\int u dv = uv - \int v du.$$

To apply integration by parts, we typically *choose* u and dv from the original integrand so that $\int v \, du$ is easier to integrate than $\int u \, dv$.

Example 1. Evaluate $\int t \sin 2t \, dt$.

If we choose u = t and $dv = \sin 2t \, dt$, then du = dt and $v = -(1/2)\cos 2t$, so that integration by parts gives

$$\int t \sin 2t \, dt = -\frac{t \cos 2t}{2} - \int \frac{-\cos 2t}{2} \, dt$$
$$= -\frac{t \cos 2t}{2} + \frac{1}{2} \int \cos 2t \, dt$$
$$= -\frac{t \cos 2t}{2} + \frac{\sin 2t}{4} + C.$$

Do NOT forgot the arbitrary constant!

This answer can be verified by differentiation:

$$\frac{d}{dt}\left[-\frac{t\cos 2t}{2} + \frac{\sin 2t}{4} + C\right] = \frac{-\cos 2t + 2t\sin 2t}{2} + \frac{2\cos 2t}{4} = t\sin 2t \checkmark.$$

Example 2. Evaluate $\int \arctan 4t \, dt$.

If we choose $u = \arctan 4t$ and dv = dt, then $du = 4dt/(1 + (4t)^2)$ and v = t, so that

$$\int \arctan 4t \, dt = t \arctan 4t - \int \frac{4t}{1 + (4t)^2} \, dt \quad [u = 1 + (4t)^2, \, du = 32tdt]$$
$$= t \arctan 4t - \frac{1}{8} \int \frac{du}{u}$$
$$= t \arctan 4t - \frac{\ln |u|}{8} + C$$
$$= t \arctan 4t - \frac{\ln |1 + (4t)^2|}{8} + C.$$

We can verify this:

$$\frac{d}{dt} \left[t \arctan 4t - \frac{\ln|1 + (4t)^2|}{8} + C \right] = \arctan 4t + t \frac{4}{1 + (4t)^2} - \frac{1}{8} \frac{32t}{1 + (4t)^2} = \arctan 4t \checkmark.$$

Example 3. Evaluate $\int (\ln x)^2 dx$.

If we let $u = \ln x$ and $dv = \ln x dx$, then du = dx/x and v = ?????. Instead we could try $u = (\ln x)^2$ and dv = dx, so that $du = 2(\ln x)/x$ and v = x, whence

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int 2\ln x \, dx = x(\ln x)^2 - 2 \int \ln x \, dx.$$

How do we integrate $\ln x$? We take $u = \ln x$ and dv = dx so that u = dx/x and v = x, whence

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C$$

Putting it all together gives

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x \, dx = x(\ln x)^2 - 2(x\ln x - x) + C.$$

We can verify this:

$$\frac{d}{dx} \left[x(\ln x)^2 - 2(x\ln x - x) + C \right] = (\ln x)^2 + 2\ln x - 2\ln x - 2 + 2 = (\ln x)^2 \checkmark.$$

Reduction Formulas for Integration. There appears in Example 3 a reduction of the power of $\ln x$ with each use of integration by parts.

Identifying these kinds of patterns gives what are called reduction formulas for integration.

Example 4. Find a reduction formula for $\int (\ln x)^n dx$ where *n* is a nonnegative integer.

If we choose $u = (\ln x)^n$ and dv = dx, then $du = nx^{-1}(\ln x)^{n-1}dx$ and v = x, so that

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

is a reduction formula for this integral.

Example 5. Find a reduction formula for $\int x^n e^x dx$ where *n* is a nonnegative integer. If we choose $u = x^n$ and $dv = e^x dx$, then $du = nx^{n-1}dx$ and $v = e^x$, so that

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

is a reduction formula for this integral.

Integration by Parts for Definite Integrals. Assuming that f'(x) and g'(x) are continuous, we can show by the Fundamental Theorem of Calculus Part II that integration by parts works for definite integrals:

$$\int_{a}^{b} f(x)g'(x) \, dx = f(x)g(x) \Big|_{a}^{b} - \int_{a}^{b} f'(x)g(x) \, dx.$$

Example 6. Evaluate $\int_{1}^{2} \frac{\ln x}{x^2} dx$.

If we choose $u = \ln x$ and $dv = x^{-2}dx$, then $du = x^{-1}dx$ and $v = -x^{-1}$, so that

$$\int_{1}^{2} \frac{\ln x}{x^{2}} dx = -\frac{\ln x}{x} \Big|_{1}^{2} + \int_{1}^{2} \frac{dx}{x^{2}}$$
$$= -\frac{\ln x}{x} \Big|_{1}^{2} - \frac{1}{x} \Big|_{1}^{2}$$
$$= -\frac{\ln 2}{2} + 0 - \frac{1}{2} + 1$$
$$= \frac{1 - \ln 2}{2}.$$

Although it may be impossible to verify this number as being correct, we can check the indefinite integral we used to do the evaluation:

$$\frac{d}{dx}\left[-\frac{\ln x}{x} - \frac{1}{x}\right] = -\frac{1 - \ln x}{x^2} + \frac{1}{x^2} = \frac{\ln x}{x^2} \checkmark$$