Math 113 Lecture #9
§7.2: Trigonometric Integrals

Evaluate Integrals of Powers of Trigonometric Functions. The use of trigonometric identities is paramount in integrating products of powers of trigonometric functions.

For nonnegative integers m and n, the integration of

$$\int \sin^m x \cos^n x \, dx$$

depends on the parity (i.e., even or odd) of m and n.

<u>Case 1: $n = 2k+1$, i.e., n is odd.</u> Here we take all but one of the cosine functions (an even power of them) and use the identity $\sin^2 x + \cos^2 x = 1$ to express each square of cosine in terms of a square of sine:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x \cos^{2k} x \cos x \, dx \\ &= \int \sin^m x (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x \big[1 - \sin^2 x\big]^k \cos x \, dx. \end{aligned}$$

The substitution $u = \sin x$, $du = \cos x dx$ gives a polynomial integrand:

$$\int \sin^m \cos^n \, dx = \int u^m [1 - u^2]^k \, du.$$

<u>Case 2: $m = 2k+1$, i.e., m is odd.</u> Here take all but one of the sine functions (an even power of them), and replace them with $1 - \cos^2 x$:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int [\sin^2 x]^k \sin x \cos^n x = \int [1 - \cos^2 x]^k \sin x \cos^n x \, dx.$$

The use of the substitution $u = \cos x$, $du = -\sin x dx$ converts the integrand into a polynomial, as in Case 1.

<u>Case 3: m and n are even.</u> Here we use the half-angle identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the even powers of sine and/or cosine into powers of cosine.

When both powers are the same (and even), the identity

$$\sin x \cos x = \frac{\sin 2x}{2}$$

may prove useful.

Example 1. Evaluate $\int \sin^4 x \cos^2 x \; dx$.

Applying the half-angle identities to this gives

$$\begin{aligned}
\int \sin^4 x \cos^2 x \; dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 \frac{1+\cos 2x}{2} \; dx \\
&= \int \frac{\left(1 - 2\cos 2x + \cos^2 2x\right)\left(1+\cos 2x\right)}{8} \; dx \\
&= \frac{1}{8}\int \left(1 + \cos 2x - 2\cos 2x - 2\cos^2 2x + \cos^2 2x + \cos^3 3x\right) \; dx \\
&= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) \; dx \\
&= \frac{1}{8}\int \left(1 - \cos 2x\right) \; dx - \frac{1}{8}\int \cos^2 2x \; dx + \frac{1}{8}\int \cos^3 2x \; dx.
\end{aligned}$$

The first integral is easy to compute, but how about the second and third integrals?

We apply the half-angle formula to the second integral, and apply Case 1 followed by the substitution $u = \sin 2x$ to convert the third integral into that of a polynomial:

$$\begin{aligned}
\int \sin^4 x \cos^2 x \; dx &= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{1}{8}\int \frac{1+\cos 4x}{2} \; dx + \frac{1}{8}\int \cos^2 2x \cos 2x \; dx \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{1}{16}\int \left(1+\cos 4x\right) \; dx + \frac{1}{8}\int \left(1-\sin^2 2x\right)\cos 2x \; dx \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{1}{16}\left(x + \frac{\sin 4x}{4}\right) + \frac{1}{16}\int \left(1-u^2\right) \; du \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} + \frac{1}{16}\left(u - \frac{u^3}{3}\right) + C \\
&= \frac{x}{16} - \frac{\sin 2x}{16} - \frac{\sin 4x}{64} + \frac{\sin 2x}{16} - \frac{\sin^3 2x}{48} + C \\
&= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C.
\end{aligned}$$

This is much harder to verify by differentiation, as it requires trigonometric identities to recover the original integrand.

For nonnegative integers m and n, the integration of

$$\int \tan^m x \sec^n x \; dx$$

depends also on the parity of m and n.

<u>Case 1: $n = 2k$, i.e., the power of secant is even.</u> Keep a factor $\sec^2 x$ and convert the rest by the trigonometric identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \tan^m x \sec^{2k} \; dx = \int \tan^m x \sec^{2(k-1)} x \sec^2 x \; dx = \int \tan^m x \left[1 + \tan^2 x\right]^{k-1} \sec^2 x \; dx.$$

The substitution $u = \tan x$, $du = \sec^2 x dx$ convert the integrand into a polynomial.

Case 2: $m = 2k + 1$, i.e., the power of tangent is odd. Here save a factor of $\sec x \tan x$ (the derivative of $\sec x$) and use $\tan^2 x = \sec^2 x - 1$ to replace all $\tan x$ by $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \, dx &= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx \\ &= \int [\sec^2 x - 1]^k \sec^{n-1} x \sec x \tan x \, dx.\end{aligned}$$

The substitution $u = \sec x$, $du = \sec x \tan x dx$ converts the integrand into a polynomial.

Example 2. Evaluate $\displaystyle\int_0^{\pi/3} \tan^5 x \sec^4 x \, dx$.

Here, the power of tangent is odd (Case 2), and the power of secant is even (Case 1). We will use Case 1 (following Case 2 is up to you):

$$\begin{aligned}\int_0^{\pi/3} \tan^5 x \sec^4 x \, dx &= \int_0^{\pi/3} \tan^5 x(1 + \tan^2 x) \sec^2 x \, dx \\ &= \int_0^{\pi/3} \left(\tan^5 x + \tan^7 x\right) \sec^2 x \, dx \quad [u = \tan x, \quad du = \sec^2 x \, dx] \\ &= \int_0^{\sqrt{3}} \left(u^5 + u^7\right) du \\ &= \left[\frac{u^6}{6} + \frac{u^8}{8}\right]_0^{\sqrt{3}} \\ &= \frac{27}{6} + \frac{81}{8} \\ &= \frac{117}{8}.\end{aligned}$$

Evaluate Other Kinds of Trigonometric Integrals.

The trigonometric identities

$$\begin{aligned}\sin A \cos B &= \frac{\sin(A - B) + \sin(A + B)}{2}, \\ \sin A \sin B &= \frac{\cos(A - B) - \cos(A + B)}{2}, \\ \cos A \cos B &= \frac{\cos(A - B) + \cos(A + B)}{2},\end{aligned}$$

have their place in computing integrals.

Example 3. For nonnegative integers m and n, compute $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$.

Applying the third identity above gives

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \int_{-\pi}^{\pi} \frac{\cos(mx - nx) + \cos(mx + nx)}{2} \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(m - n)x + \cos(m + n)x\right) dx.\end{aligned}$$

What happens next depends on m and n: if $m = n$, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx \; dx &= \frac{1}{2} \int_{-\pi}^{\pi} \big(\cos(m-n)x + \cos(m+n)x \big) \; dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \big(1 + \cos(2mx) \big) \; dx \\
&= \frac{1}{2} \left[x + \frac{\sin 2mx}{2m} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2} \big[\pi - (-\pi) \big] \\
&= \pi.
\end{aligned}$$

On the other hand, if $m \neq n$, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx \; dx &= \frac{1}{2} \int_{-\pi}^{\pi} \big(\cos(m-n)x + \cos(m+n)x \big) \; dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

Apply Trigonometric Integrals to Finite Fourier Series.

A finite Fourier series is

$$f(x) = \sum_{n=1}^{N} \big(a_n \sin nx + b_n \cos nx \big).$$

Suppose for now that $a_n = 0$, i.e., f is a finite Fourier cosine series,

$$f(x) = \sum_{n=1}^{N} b_n \cos nx.$$

What is the relationship between f and the Fourier cosine coefficients $b_1, \ldots, b_N$?

For $m = 1, \ldots, N$, integration gives the relationship:

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \; dx &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} b_n \cos nx \right) \cos mx \; dx \\
&= \frac{1}{\pi} \sum_{n=1}^{N} b_n \int_{-\pi}^{\pi} \cos mx \cos nx \; dx.
\end{aligned}$$

These integrals are zero when $m \neq n$, and is π when $m = n$ (as shown in Example 3); so

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx = \frac{1}{\pi} \big(b_n \pi \big) = b_n.$$