Math 113 Lecture #10
§7.3: Trigonometric Substitutions

Inverse Substitutions by Trigonometric (Hyperbolic) Functions. Reversing the Substitution Rule sometimes leads to simplier integrals: if $x = g(t)$ for g invertible and differentiable, then $dx = g'(t)dt$ and

$$\int f(x)\ dx = \int f(g(t))g'(t)\ dt.$$

This kind of *inverse substitution* can give $f(g(t))g'(t)$ as simpler to integrate than $f(x)$.

The choice of g as a trigonometric or hyperbolic trigonometric function can eliminate square roots from the integrand.

Example 1. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\ dx$.

We can use the trigonometric identity $\sec^2\theta - 1 = \tan^2\theta$ to choose an inverse substitution:

$$x = 3\sec\theta, \quad dx = 3\sec\theta\tan\theta\ d\theta.$$

Why the factor of 3?

With this inverse substitution, the integral becomes

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-9}}\ dx &= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}\ d\theta \\ &= \int \frac{\sec\theta\tan\theta}{9\sec^2\theta\sqrt{\tan^2\theta}}\ d\theta \\ &= \int \frac{\sec\theta\tan\theta}{9\sec^2\theta|\tan\theta|}\ d\theta.\end{aligned}$$

Remember that $\sqrt{y^2} = |y|$ in general, and that $\sqrt{y^2} = y$ only if we known that $y \geq 0$.

So we ASSUME that $\tan\theta \geq 0$, so that $|\tan\theta| = \tan\theta$, i.e., that $0 < \theta < \pi/2$.

With the assumption, the integral becomes

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-9}}\ dx &= \frac{1}{9}\int \frac{1}{\sec\theta}\ d\theta \\ &= \frac{1}{9}\int \cos\theta\ d\theta \\ &= \frac{\sin\theta}{9} + C.\end{aligned}$$

We must express this indefinite integrals in terms of the original variable x, but how?

The function sec in $x = 3\sec\theta$ is invertible on $(0, \pi/2)$, and so

$$\theta = \sec^{-1}(x/3).$$

Thus the indefinite integral is

$$\int \frac{1}{x^2\sqrt{x^2-9}}\ dx = \frac{\sin\left(\sec^{-1}(x/3)\right)}{9} + C.$$

The composition of sine with the inverse of secant is messy. Can it be simplified?

Yes, it can with the use of a right-angle triangle: one angle is θ which lies between 0 and $\pi/2$, and since $\sec\theta = x/3$, the side of the triangle adjacent to the angle θ has length 3 and the hypothenuse has length x.

The Pythagorean Theorem then gives the length of the side opposite θ as $\sqrt{x^2-9}$, and so

$$\sin(\sec^{-1}(x/3)) = \sin\theta = \frac{\sqrt{x^2-9}}{x}.$$

The indefinite integral is then

$$\int \frac{1}{x^2\sqrt{x^2-9}}\ dx = \frac{\sqrt{x^2-9}}{9x} + C.$$

We can (and should) verify this (especially after all the work we did to get it):

$$\begin{aligned}
\frac{d}{dx}\frac{\sqrt{x^2-9}}{9x} &= \frac{(1/2)(x^2-9)^{-1/2}(2x)(9x) - 9\sqrt{x^2-9}}{81x^2} \\
&= \frac{x^2(x^2-9)^{-1/2} - \sqrt{x^2-9}}{9x^2}\ \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}} \\
&= \frac{x^2-(x^2-9)}{9x^2\sqrt{x^2-9}} \\
&= \frac{1}{x^2\sqrt{x^2-9}}\ \checkmark.
\end{aligned}$$

Example. Evaluate $\displaystyle\int x^3\sqrt{9-x^2}\ dx$.

The trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ suggests the inverse substitution of

$$x = 3\sin\theta, \quad dx = 3\cos\theta\ d\theta.$$

For θ in $(0,\pi/2)$, the indefinite integral becomes

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\ dx &= \int 27\sin^3\theta\sqrt{9-9\sin^2\theta}\ (3\cos\theta)\ d\theta \\
&= 243\int \sin^3\theta\cos^2\theta\ d\theta.
\end{aligned}$$

With the power of sine being odd, we use the substitution $u=\cos\theta$, $du = -\sin\theta d\theta$ to convert the integrand into a polynomial:

$$
\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= 243\int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta \\
&= -243\int (1-u^2)u^2\,du \\
&= -243\int \left(u^2-u^4\right)\,du \\
&= -243\left[\frac{u^3}{3}-\frac{u^5}{5}\right]+C.
\end{aligned}
$$

Since $x = 3\sin\theta$ and θ is in $(0, \pi/2)$ where sin is invertible, we get $\theta = \sin^{-1}(x/3)$. Since $u = \cos\theta$, we get

$$
\int x^3\sqrt{9-x^2}\,dx = -243\left[\frac{\cos^3(\sin^{-1}(x/3))}{3}-\frac{\cos^5(\sin^{-1}(x/3))}{5}\right]+C.
$$

Again, by a right-angled triangle with θ as one angle, x as the side opposite θ, and 3 as the hypothenuse, we get that

$$
\cos(\sin^{-1}(x/3)) = \frac{\sqrt{9-x^2}}{3}.
$$

Thus the indefinite integral is

$$
\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= -243\left[\frac{(9-x^2)^{3/2}}{81}-\frac{(9-x^2)^{5/2}}{243\times 5}\right]+C \\
&= -3(9-x^2)^{3/2}+\frac{(9-x^2)^{5/2}}{5}+C.
\end{aligned}
$$

We may think that this does not "look" right, but we can tell for sure by verification:

$$
\begin{aligned}
\frac{d}{dx}\left[-3(9-x^2)^{3/2}+\frac{(9-x^2)^{5/2}}{5}+C\right] &= -\frac{9}{2}\sqrt{9-x^2}(-2x)+\frac{1}{2}(9-x^2)^{3/2}(-2x) \\
&= 9x\sqrt{9-x^2}-x(9-x^2)^{3/2} \\
&= x\sqrt{9-x^2}\big(9-(9-x^2)\big) \\
&= x^3\sqrt{9-x^2}\checkmark\ .
\end{aligned}
$$

Evaluation of Definite Integrals by Inverse Substitutions.

When we deal with definite integrals, we do not have to undo all the changes we made along the way.

Example 3. Evaluate $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$.

The trigonometric identity $1+\tan^2\theta = \sec^2\theta$ suggest the inverse substitution

$$
x = \tan\theta, \quad dx = \sec^2\theta\,d\theta.
$$

Here the limits of integration become $0 = \tan u$ or $u = 0$, and $1 = \tan u$ or $u = \pi/4$, so that the definite integral becomes

$$\begin{aligned}\int_0^1 \sqrt{x^2+1}\ dx &= \int_0^{\pi/4} \sqrt{\tan^2\theta + 1}\ \sec^2\theta\ d\theta \\ &= \int_0^{\pi/4} \sec^3\theta\ d\theta.\end{aligned}$$

With the power of secant being odd, there is no trigonometric identity that will help us here.

Instead we opt for an integration by parts approach: with

$$u = \sec\theta, \quad dv = \sec^2\theta\ d\theta, \quad du = \sec\theta\tan\theta\ d\theta, \quad v = \tan\theta,$$

the definite integral becomes

$$\begin{aligned}\int_0^{\pi/4} \sec^3\theta\ d\theta &= \sec\theta\tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4} \sec\theta\tan^2\theta\ d\theta \\ &= \frac{2}{\sqrt{2}} - \int_0^{\pi/4} \sec\theta(\sec^2\theta - 1)\ d\theta \\ &= \frac{2}{\sqrt{2}} - \int_0^{\pi/4} \sec^3\theta\ d\theta + \int_0^{\pi/4} \sec\theta\ d\theta.\end{aligned}$$

Combining the two integrals involving the cube of secant gives

$$\begin{aligned}\int_0^{\pi/4} \sec^3\theta\ d\theta &= \frac{1}{\sqrt{2}} + \frac{1}{2}\int_0^{\pi/4} \sec\theta\ d\theta \\ &= \frac{1}{\sqrt{2}} + \frac{1}{2}\Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{2}\left[\ln\left|\frac{2}{\sqrt{2}} + 1\right| - \ln|1+0|\right] \\ &= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln(1+\sqrt{2}).\end{aligned}$$

Along the way, we learned that

$$\int \sec^3\theta\ d\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C,$$

from using integration by parts and

$$\int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C.$$