Math 113 Lecture #11
§7.4: Partial Fractions, Part I

Polynomial Division.
Integration of a rational function requires techniques from algebra to convert the rational function into a form that is easily integrated.

If the degree of the polynomial $P(x)$ in the numerator is the same or bigger than the degree of the polynomial $Q(x)$ in the denominator, the first bit of algebra to be done is polynomial division:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ and $R(x)$ are polynomial with the degree of $R(x)$ being strictly smaller than that of $Q(x)$.
There is no problem in integrating $S(x)$ since it is a polynomial.

Example 1. Find $S(x)$ and $R(x)$ by applying polynomial division to

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^2}{x+4}.$$

The answer is

$$\frac{x^2}{x+4} = x - 4 + \frac{16}{x+4}.$$

This can (and should) be checked:

$$x - 4 + \frac{16}{x+4} = \frac{(x-4)(x+4)+16}{x+4} = \frac{x^2-16+16}{x+4} = \frac{x^2}{x+4}\checkmark .$$

Thus $S(x) = x - 4$ and $R(x) = 16$.

Example 2. Use polynomial division to simplify

$$\frac{x^3}{x^2+2x+1}.$$

The simplification obtained is

$$\frac{x^3}{x^2+2x+1} = x - 2 + \frac{3x+2}{x^2+2x+1}.$$

Partial Fractions with Distinct Linear Factors.
Now we suppose that in the rational function $R(x)/Q(x)$, with the degree of R strictly less than that of $Q(x)$, that $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k),$$

where k is the degree of Q.

In this case there is a partial fraction theorem that states that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

The point of this is that each of the terms of the partial fraction decomposition (the terms on the right side) is easily integrated.

Example 1 Continued. Evaluate $\displaystyle\int \frac{x^2}{x+4}\,dx$.

We saw before that by polynomial division the integrand simplifies where the rational part has one linear factor in the denominator:

$$\begin{aligned}\int \frac{x^2}{x+4}\,dx &= \int \left(x - 4 + \frac{16}{x+4}\right)\,dx \\ &= \frac{x^2}{2} - 4x + 16\ln|x+4| + C.\end{aligned}$$

Example 3. Evaluate $\displaystyle\int \frac{x-2}{x^2-9x-10}\,dx$.

The denominator factors into linear terms:

$$x^2 - 9x - 10 = (x-10)(x+1).$$

The partial fraction decomposition of the integrand is then

$$\frac{x-2}{(x-10)(x+1)} = \frac{A}{x-10} + \frac{B}{x+1}.$$

The values of the constants A and B are found by equating the numerators (after finding the common denominator):

$$x - 2 = A(x+1) + B(x-10).$$

Multiplying this out gives two linear equations in two unknowns, which can be solved for A and B.

But with distinct linear factors there is a short-cut: the partial decomposition works for all values of x, so it works for some values of x, say $x=-1$ and $x=10$.

Why are these values of x singled out? Because each eliminates one of the unknown constants from the equation:

$$\begin{aligned}x=-1:&\ \ (-1)-2 = A(-1+1) + B(-1-10) = -11B \quad\Rightarrow\quad B = 3/11,\\ x=10:&\ \ 10-2 = A(10+1) + B(10-10) = 11A \quad\Rightarrow\quad A = 8/11.\end{aligned}$$

The partial fraction decomposition is

$$\frac{x-2}{x^2-9x-10} = \frac{8/11}{x-10} + \frac{3/11}{x+1}.$$

You can (and should) check this:

$$\frac{8/11}{x-10}+\frac{3/11}{x+1}=\frac{(8/11)(x+1)+(3/11)(x-10)}{(x-10)(x+1)}=\frac{x-22/11}{x^2-9x-10}=\frac{x-2}{x^2-9x-10}.$$

The indefinite integral is

$$\begin{aligned}\int \frac{x-2}{x^2-9x-10}\,dx &= \int\left(\frac{8/11}{x-10}+\frac{3/11}{x+1}\right)\,dx\\ &= \frac{8\ln|x-10|}{11}+\frac{3\ln|x+1|}{11}+C.\end{aligned}$$

Partial Fractions with Linear Factors, Some Repeated.

When a linear factor is repeated, the form of the partial fraction decomposition changes.

If in $R(x)/Q(x)$, the polynomial $Q(x)$ factors, say for $n \geq 2$, as

$$(a_1x+b_1)^n(a_2x+b_2)\cdots(a_kx+b_k)$$

then the partial fraction decomposition is

$$\begin{aligned}\frac{R(x)}{Q(x)} &= \frac{A_{11}}{a_1x+b_1}+\frac{A_{12}}{(a_1x+b_1)^2}+\cdots+\frac{A_{in}}{(a_1x+b_1)^n}\\ &\quad+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}.\end{aligned}$$

If more than one linear factor is repeated, each repeated factor corresponds to the same pattern given above for one repeated linear factor.

Example 2 Continued. Evaluate $\displaystyle\int \frac{x^3}{x^2+2x+1}\,dx.$

We saw before that by polynomial division, the integrand becomes

$$\frac{x^3}{x^2+2x+1}=x-2+\frac{3x+2}{x^2+2x+1}.$$

The denominator factors into repeated linear factors:

$$x^2+2x+1=(x+1)^2.$$

The partial fraction decomposition is

$$\frac{3x+2}{x^2+2x+1}=\frac{A}{x+1}+\frac{B}{(x+1)^2}.$$

Equating the numerators of both sides (after finding the common denominator) gives

$$3x+2=A(x+1)+B=Ax+A+B.$$

From this A must be 3 and so $2=A+B=3+B$ requires that $B=-1$.

The partial fraction decomposition is

$$\frac{3x+2}{x^2+2x+1} = \frac{3}{x+1} + \frac{-1}{(x+1)^2}.$$

You can (and should) check this:

$$\frac{3}{x+1} + \frac{-1}{(x+1)^2} = \frac{3(x+1)-1}{(x+1)^2} = \frac{3x+3-1}{(x+1)^2} = \frac{3x+2}{x^2+2x+1}.$$

Now the indefinite integral can be evaluated:

$$\begin{aligned}
\int \frac{x^3}{x^2+2x+1}\,dx &= \int \left(x - 2 + \frac{3x+2}{x^2+2x+1}\right)\,dx \\
&= \int (x-2)\,dx + \int \left(\frac{3}{x+1} + \frac{-1}{(x+1)^2}\right)\,dx \\
&= \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C.
\end{aligned}$$