Math 113 Lecture #12
§7.4: Partial Fractions, Part II

Partial Fractions with Distinct Irreducible Quadratic Factors. Suppose for a rational function $R(x)/Q(x)$, with the degree of R strictly smaller than the degree of Q.

Suppose further that the factorization of Q has an quadratic term $ax^2 + bx + c$, where the discriminant $b^2 - 4ac < 0$, i.e., it has complex roots, or is irreducible.

The partial fraction theory states that the decomposition for an irreducible quadratic factor has the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example 1. Evaluate $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$.

The discriminant of the quadratic denominator is $4 - 20 = -16 < 0$, and so it is irreducible.

The rational function is already in the partial fraction form.

So what do we do now?????

We complete the square of the denominator first:

$$x^2 + 2x + 5 = x^2 + 2x + 1 - 1 + 5 = (x+1)^2 + 4.$$

We then split-up the indefinite integral:

$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}\,dx &= \int \frac{x+4}{(x+1)^2+4}\,dx \\ &= \int \frac{x+1}{(x+1)^2+4}\,dx + \int \frac{3}{(x+1)^2+4}\,dx.\end{aligned}$$

Now we apply Inverse Substitution to each indefinite integral: we use

$$2u = x + 1, \quad 2du = dx$$

for both, which then become

$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}\,dx &= \int \frac{4u}{(2u)^2+4}\,du + \int \frac{6}{(2u)^2+4}\,du \\ &= \frac{1}{2}\int \frac{2u}{u^2+1}\,du + \frac{3}{2}\int \frac{1}{u^2+1}\,du.\end{aligned}$$

The first integral is a natural logarithm, and the second integral is an arctan:

$$\int \frac{x+4}{x^2+2x+5}\,dx = \frac{\ln|u^2+1|}{2} + \frac{3}{2}\arctan(u) + C.$$

Undoing the reverse substitution $2u = x + 1$ gives

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+5}\,dx &= \frac{1}{2}\ln\left|\left(\frac{x+1}{2}\right)^2 + 1\right| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C \\
&= \frac{1}{2}\ln\left(\frac{x^2+2x+1}{4} + 1\right) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C \\
&= \frac{1}{2}\ln\left(\frac{x^2+2x+1+4}{4}\right) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C \\
&= \frac{1}{2}\ln\left(x^2+2x+5\right) - \frac{\ln 4}{2} + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C \\
&= \frac{1}{2}\ln\left(x^2+2x+5\right) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

The constant $(\ln 4)/2$ has been absorbed into the arbitrary constant C.

Example 2. Evaluate $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}\,dx.$

Because the factors in the denominator are irreducible and distinct, the partial fraction decomposition is

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}.$$

The constants A, B, C, D must satisfy

$$1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1).$$

Unfortunately, there are no real zeroes to use here to quickly find A, B, C, D.

Multiplying the products gives

$$1 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D.$$

For these two polynomials to be equal requires that the coefficients of like degree be the same:

$$\begin{aligned}
x^3 &: \; A + C = 0, \\
x^2 &: \; B + D = 0, \\
x &: \; 4A + C = 0, \\
1 &: \; 4B + D = 1.
\end{aligned}$$

Solving the first and third equations in A and C simultaneously gives $A = 0$ and $C = 0$.

Solving the second and fourth equations in B and D simultaneously gives

$$1 = 4B + D = 4B - B = 3B \quad\Rightarrow\quad B = 1/3 \text{ and } D = -1/3.$$

So the partial fraction decomposition is

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1/3}{x^2+1} + \frac{-1/3}{x^2+4}.$$

We can make use of the integral formula

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

to carry out the integration:

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{1}{3}\arctan(x) - \frac{1}{6}\arctan\left(\frac{x}{2}\right) + C.$$

Partial Fractions with Repeated Irreducible Quadratic Factors.

When there are repeated irreducible quadratic factors in the denominator of a rational function, the partial fraction form for each repeated irreducible quadratic factor is

$$\frac{R(x)}{(ax^2+bx+c)^k} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}.$$

Example 3. Evaluate $\displaystyle\int \frac{dx}{x(x^2+4)^2}$.

The partial fraction decomposition for the integrand is

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}.$$

The five constants here satisfy

$$\begin{aligned} 1 &= A(x^2+4)^2 + (Bx+C)(x)(x^2+4) + (Dx+E)(x) \\ &= A(x^4+8x^2+16) + (Bx^2+Cx)(x^2+4) + Dx^2 + Ex \\ &= Ax^4 + 8Ax^2 + 16A + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex. \end{aligned}$$

Equating coefficients of like power gives

$$A = \frac{1}{16},\quad 4C+E=0,\quad 8A+4B+D=0,\quad C=0,\quad A+B=0.$$

From these we get

$$E=0,\quad B=-\frac{1}{16},\quad D=-8A-4B=-\frac{1}{2}+\frac{1}{4}=-\frac{1}{4}.$$

The partial fraction decomposition is

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1/16}{x} + \frac{-x/16}{x^2+4} - \frac{x/4}{(x^2+4)^2}.$$

Now we can carry out the integration:

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\ln|x|}{16} - \frac{\ln(x^2+4)}{32} + \frac{1}{8}\left(\frac{1}{x^2+4}\right) + C.$$

Rationalizing Substitutions. Sometimes non-rational functions can be transformed into rational functions by substitution.

In particular, when there are non-rational functions like $x^{1/n}$ in the integrand, the inverse substitution $x = u^n$ might rationalize the integrand.

Example 4. Evaluate $\displaystyle\int_0^1 \frac{1}{1+x^{1/3}}\,dx$.

We try the substitution

$$x = u^3, \quad du = 3u^2\;dx.$$

This gives

$$\int \frac{1}{1+x^{1/3}}\;dx = \int \frac{3u^2}{1+u}\;du.$$

We now have a rational function, and proceed to integrate it.