Math 113 Lecture #13
§7.5: Strategy for Integration

We have learned many techniques of integration, but unlike differentiation, there may not be a sense of which ones we apply and when we apply them.

It takes much practice to establish the art of integration.

Here are steps you can try when integrating.

Step 1. Algebraically simplify the integrand, if possible. This may reveal which techniques of integration will work.

Step 2. Look for an obvious Substitution. If there is an obvious substitution, then the substitution will simplify the integrand to one that can be integrated or to one for which the next step of integration is obvious.

Step 3. Classify the integrand according to its form. The form of the integrand gives a clue as to which technique of integration might work. The basic forms are trigonometric, rational, and radical.

Step 4. Try Again. Sometimes one approach will fail, so try another approach. Keep trying – you will get it eventually. Try integration by parts, manipulate the integrand, or relate the integrand to known problems.

Example 1. Evaluate $\displaystyle\int \exp(x + e^x) \, dx$.

This looks messy, but can we simplify the integrand at all? Well,

$$\int \exp(x + e^x) \, dx = \int e^x \exp(e^x) \, dx.$$

This simplification suggests that we try the substitution,

$$u = e^x, \quad du = e^x \, dx.$$

The indefinite integral becomes

$$\int \exp(x + e^x) \, dx = \int e^u \, du = e^u + C = \exp(e^x) + C.$$

We can verify that we found the correct indefinite integral:

$$\frac{d}{dx}\big(\exp(e^x)\big) = \exp(e^x)e^x = \exp(x + e^x)\checkmark \ .$$

Example 2. Evaluate $\displaystyle\int \frac{x}{\sqrt{3 - x^4}} \, dx$.

There does not appear to be an algebraic simplification of the integrand here.

So, instead we try a substitution, say

$$u = x^2, \quad du = 2x \, dx.$$

The indefinite integral becomes

$$\int \frac{x}{\sqrt{3-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{3-u^2}}\,du.$$

We can now apply the trigonometric substitution

$$u = \sqrt{3}\,\sin\theta, \quad du = \sqrt{3}\,\cos\theta\,d\theta.$$

The indefinite integral is now

$$\begin{aligned}
\int \frac{x}{\sqrt{3-x^4}}\,dx &= \frac{1}{2}\int \frac{\sqrt{3}\,\cos\theta}{\sqrt{3-3\sin^2\theta}}\,d\theta \\
&= \frac{1}{2}\int d\theta \\
&= \frac{\theta}{2} + C \\
&= \frac{1}{2}\arcsin\left(\frac{u}{\sqrt{3}}\right) + C \\
&= \frac{1}{2}\arcsin\left(\frac{x^2}{\sqrt{3}}\right) + C.
\end{aligned}$$

Example 3. Evaluate $\int x^5 e^{-x^3}\,dx$.

There does not appear to be an algebraic simplification of the integrand, nor a substitution that simplifies the integrand.

So we try integration by parts:

$$u = x^3, \quad dv = x^2 e^{-x^3}, \quad du = 3x^2 dx, \quad v = -(e^{-x^3})/3.$$

The indefinite integral becomes

$$\int x^5 e^{-x^3}\,dx = -\frac{x^3 e^{-x^3}}{3} + \int x^2 e^{-x^3}\,dx.$$

The last integral can be integrated by substitution:

$$u = -x^3, \quad du = -3x^2\,dx.$$

So the indefinite integral is

$$\begin{aligned}
\int x^5 e^{-x^3}\,dx &= -\frac{x^3 e^{-x^3}}{3} - \frac{1}{3}\int e^u\,du \\
&= -\frac{x^3 e^{-x^3}}{3} - \frac{e^u}{3} + C \\
&= -\frac{x^3 e^{-x^3}}{3} - \frac{e^{-x^3}}{3} + C.
\end{aligned}$$

Example 4. Evaluate $\int x(x+1)^{1/3}\, dx$.

It looks like there may be a rationalizing substitution here:

$$u^3 = x + 1, \quad 3u^2 du = dx.$$

The indefinite integral is

$$\begin{aligned}
\int x(x+1)^{1/3}\, dx &= 3 \int (u^3 - 1)u^3\, du \\
&= 3 \int \left(u^6 - u^3\right)\, du \\
&= 3\left[\frac{u^7}{7} - \frac{u^4}{4}\right] + C \\
&= \frac{3}{7}(x+1)^{7/3} - \frac{3}{4}(x+1)^{4/3} + C.
\end{aligned}$$

Example 5. Evaluate $\int \dfrac{xe^x}{\sqrt{1+e^x}}\, dx$.

The presence of x in the integrand suggests integration by parts:

$$u = x, \quad dv = \frac{e^x dx}{\sqrt{1+e^x}}, \quad du = dx, \quad v = 2\sqrt{1+e^x}.$$

Thus the indefinite integral is

$$\int \frac{xe^x}{\sqrt{1+e^x}}\, dx = 2x\sqrt{1+e^x} - 2 \int \sqrt{1+e^x}\, dx.$$

For the last integral we try a rationalizing substitution:

$$u^2 = 1 + e^x, \quad 2u\,du = e^x dx \ \text{ or } \ \frac{2u\,du}{u^2 - 1} = dx.$$

The indefinite integral is then

$$\begin{aligned}
\int \frac{xe^x}{\sqrt{1+e^x}}\, dx &= 2x\sqrt{1+e^x} - 2 \int \frac{2u^2 du}{u^2 - 1} \\
&= 2x\sqrt{1+e^x} - 4 \int \frac{u^2}{u^2 - 1}\, du \\
&= 2x\sqrt{1+e^x} - 4 \int \left(1 - \frac{1}{1 - u^2}\right)\, du \\
&= 2x\sqrt{1+e^x} - 4u + 4 \int \left\{\frac{1/2}{1 - u} + \frac{1/2}{1 + u}\right\} \\
&= 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\ln\left|1 - \sqrt{1+e^x}\right| + 2\ln\left|1 + \sqrt{1+e^x}\right| + C.
\end{aligned}$$