Math 113 Lecture #14
§7.7: Approximate Integration, Part I

There are two main situations in which evaluating a definite integral by the Fundamental Theorem of Calculus (i.e., finding an antiderivate) is impossible.

The first is when the antiderivative is not a "nice" function, like for

$$\int_0^1 e^{x^2}\,dx.$$

The second situation is when the function has no known formula, and a sampling of values of the function have been determined by measurements.

In both case, we need to find approximations of the definite integrals, and this is done by numerical algorithms.

Any Riemann sum can be used as an approximation for a definite integral.

This is of course contingent on the ASSUMPTION that the function being integrated is integrable.

Most functions we would be interested in integrating satisfy this assumption.

The Left Endpoint, Right Endpoint, and Midpoint Rules (Again). Of the Riemann sums used for approximate integration, the left, right, and midpoint rules are the most common.

Let f be integrable on $[a, b]$.

For an positive integer n, we let $\Delta x = (b - a)/n$ and divide $[a, b]$ into subintervals of equal length with endpoints $x_i = a + i\Delta x$.

The *left endpoint approximation* and its error to $\displaystyle\int_a^b f(x)\,dx$ are

$$L_n \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad E_L = \int_a^b f(x)dx - L_n,$$

the *right endpoint approximation* and its error are

$$R_n \approx \sum_{i=1}^{n} f(x_i)\Delta x, \quad E_R = \int_a^b f(x)dx - R_n$$

and for $\bar{x}_i = (x_{i-1} + x_i)/2$, the *midpoint approximation* and its error are

$$M_n \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x, \quad E_M = \int_a^b f(x)dx - M_n.$$

Example 1. Let us investigate these approximations and their errors for

$$\int_0^1 xe^x\,dx = \Big[xe^x - e^x\Big]_0^1 = 1.$$

We will do this for $n = 5, 10, 20$.

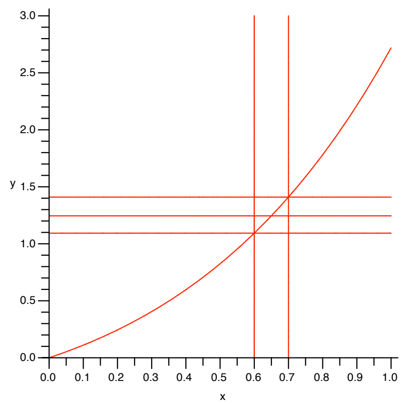
We have arranged the “data” to six digits in the following table.

n	L_n	E_L	R_n	E_R	M_n	E_M
5	0.742943	0.257057	1.28660	-.286600	0.992621	0.007379
10	0.867782	0.132218	1.13961	-.139610	0.998152	0.001848
20	0.932967	0.067033	1.06888	-0.06888	0.999538	0.000462

By far the midpoint approximation is the best of these three.

Why would the midpoint approximation be better than the left endpoint and the right endpoint approximations?

This can be answered by considering the typical graph of f on the interval $[x_{i-1}, x_i]$.



The function $f(x) = xe^x$ is increasing on $[0, 1]$, so that the left endpoint approximation under estimates the integral, the right endpoint approximation over estimates the integral.

But the midpoint approximation gives a better estimate of the integral.

Error Estimate for Midpoint Approximation Through advanced numerical analysis, we can obtain an estimate of the error of the midpoint approximation.

We assume that f is twice differentiable on an open interval containing $[a, b]$ and that there is a positive constant K for which

$$|f''(x)| \leq K \text{ for all } x \text{ in } [a, b].$$

Then we have

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Notice that as we increase n the midpoint approximation gets closer to the actual value of the integral.

And notice further that the error estimate does not require the value of integral!

All we need to know up front is the conditions on the second derivative of the function we want to integrate.

We can choose the number n before computing the midpoint approximation to get the midpoint approximation as close as we want to the actual value of the integral.

Example 1 Continued. What value of n gives a midpoint approximation of

$$\int_0^1 xe^x \, dx$$

that is within 0.0001 of the actual value?

To use the error estimate we need to know

$$f''(x) = \frac{d}{dx}\left(xe^x + e^x\right) = xe^x + 2e^x.$$

The maximum value K of the increasing f'' is obtained at $x = b$:

$$K = e + 2e = 3e.$$

Plugging this into the error estimate for the midpoint approximation gives an inequality in n:

$$|E_M| \leq \frac{3e(1-0)^3}{24n^2} = \frac{3e}{24n^2} < 0.0001.$$

Solving for n gives

$$n \geq \sqrt{\frac{3e}{24(0.0001)}} \approx 58.2911.$$

So we need at least $n = 59$. Indeed,

$$M_{59} = 0.999947 \text{ and } E_M = 0.000053.$$

The Trapezoidal Rule.

Averaging the left endpoint and the right endpoint approximations results in the trapezoidal rule for approximation:

$$\begin{aligned}
T_n &= \frac{L_n + R_n}{2} \\
&= \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right] \\
&= \frac{\Delta x}{2}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + f(x_1) + f(x_2) + \cdots + f(x_n)\right] \\
&= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].
\end{aligned}$$

Geometrically, the trapezoidal approximation adds the areas of the trapezoids over the subintervals, each of whose tops are the straight line from the point $(x_{i-1}, f(x_{i-1}))$ on the graph to the point $(x_i, f(x_i))$ on the graph.

Can you visualize this?

The error of the trapezoidal rule from the actual integral is

$$E_T = \int_a^b f(x)\ dx - T_n.$$

If $f''(x)$ exists on an open interval containing $[a, b]$ and there is a positive constant K such that

$$|f''(x)| \leq K \text{ for all } x \text{ in } [a, b],$$

then we have the error estimate for the trapezoidal rule:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

In comparison the trapezoidal approximation is not as good as the midpoint approximation.

Example 1 Continued (Again). Here are the midpoint and trapezoidal approximations and their errors for

$$\int_0^1 xe^x\ dx = 1.$$

n	M_n	E_M	T_n	E_T
5	0.992621	0.007379	1.01477	-0.01477
10	0.998152	0.001848	1.00370	-0.00369
20	0.999538	0.000462	1.00092	-0.00092

To get a trapezoidal approximation that is within 0.0001 of the actual integral requires that n satisfy

$$|E_T| \leq \frac{3e}{12n^2} \leq 0.0001 \quad \Rightarrow \quad n \geq 82.35.$$

So we need as least $n = 83$. Indeed,

$$T_{83} = 1.00005 \text{ and } E_T = -0.00005.$$