Simpson's Rule. The Midpoint and Trapezoidal Rules use straight lines to top the approximating areas for the value of

$$\int_a^b f(x)\ dx.$$

Simpson's Rule uses parabolas to top the approximating areas.

Again we divide $[a, b]$ into equal subintervals of length $\Delta x = (b - a)/n$ and endpoints $x_i = a + i\Delta x$.

Unlike the previous rules, we now ASSUME that n is even.

For $y_i = f(x_i)$, the point $P_i = (x_i, y_i)$ lies on the graph of f.

We will find the parabola that passes through three consecutive point P_i, P_{i+1}, and P_{i+2}.

This parabola approximates the graph of f over the interval $[x_i, x_{i+2}]$.

The point of doing this is that integrating the approximating parabola is much easier and quicker than integrating f over the interval.

Before we find the approximating parabola, let us consider the integral of the parabola $Ax^2 + Bx + C$ over the interval $[-h, h]$ for $h > 0$:

$$
\begin{aligned}
\int_{-h}^{h} \left(Ax^2 + Bx + C\right)\ dx &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h} \\
&= \frac{Ah^3}{3} + \frac{Bh^2}{2} + Ch - \left(\frac{A(-h)^3}{3} + \frac{B(-h)^2}{2} + C(-h)\right) \\
&= \frac{Ah^3}{3} + \frac{Bh^2}{2} + Ch + \frac{Ah^3}{3} - \frac{Bh^2}{2} + Ch \\
&= \frac{2Ah^3}{3} + 2Ch \\
&= \frac{h}{3}\left(2Ah^2 + 6C\right).
\end{aligned}
$$

The values of A, B, and C are determined by the three poins $(-h, y_0)$, $(0, y_1)$, and (h, y_2) that the parabola passes through:

$$
\begin{aligned}
y_0 &= A(-h)^2 + B(-h) + C = Ah^2 - Bh + C, \\
y_1 &= C, \\
y_2 &= Ah^2 + Bh + C.
\end{aligned}
$$

Now there is a nice connection between the values of y_0, y_1, and y_2, and the integral of the parabola: since

$$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C,$$

then

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{h}{3}\left(2Ah^2 + 6C\right).$$

This means that the integral of the approximating parabola can be computed using the three y values of the points through which the parabola passes.

Now shifting the parabola horizontal does not change the value of its integral.

So we can apply that same formula to the parabola passing through any three consecutive points P_i, P_{i+1}, and P_{i+2} on the graph of f.

Thus the integral of the parabola passing through P_0, P_1, and P_2 is

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

The integral of the parabola passing through P_2, P_3, and P_4 is

$$\frac{h}{3}\left(y_2 + 4y_3 + y_4\right).$$

Since we assumed that n is even, this process with terminate with the parabola passing through P_{n-2}, P_{n-1}, and P_n:

$$\frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right).$$

In all of these, $h = \Delta x$, and we get Simpson's Rule:

$$\int_a^b f(x)\ dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

The error for Simpson's Rule is

$$E_S = \int_a^b f(x)\ dx - S_n.$$

If f is four times differentiable and there is a constant K such that

$$|f^{(4)}(x)| \le K \text{ for all } x \text{ in } [a, b],$$

then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$

This is a better estimate that that for the Midpoint and Trapezoidal Rules.

Example 1. We saw last time that

$$\int_0^1 xe^x\ dx = 1.$$

Here are approximations and errors for the Trapezoidal, Midpoint, and Simpson's Rules.

n	T_n	E_T	M_n	M_T	S_n	E_S
4	1.02306	-0.02306	0.988484	0.011516	1.00017	-0.00017
10	1.00370	-0.00370	0.998152	0.001848	1.00000	0.00000
20	1.00092	-0.00092	0.999538	0.000462	1.00000	0.00000

We can see that Simpson's Rule is far superior to the Trapezoidal and Midpoint Rules.

Example 2. Use Simpson's rule with $n = 10$ to estimate the value of

$$\int_{-1}^{1} \exp(-x^2/2) \ dx.$$

What is the error this estimate?

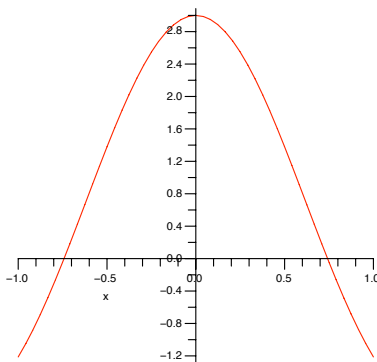Simpson's rule with $n = 10$ gives

$$S_{10} = 1.711270662.$$

The fourth derivative of $f(x) = \exp(-x^2/2)$ is

$$f^{(4)}(x) = 3\exp(-x^2/2) - 6x^2 \exp(-x^2/2) + x^4 \exp(-x^2/2).$$

Here is the graph of the fourth derivative of f.

The choice of $K = 3$ satisfies $|f^{(4)}(x)| \leq K$ for all x in $[-1, 1]$.

The error of the estimate S_{10} is

$$|E_S| \leq \frac{3\big(1 - (-1)\big)^5}{180(10)^4} = \frac{1}{18750} = 0.000053333....$$

What choice of n would guarantee that S_n is within 0.000001 of the actual value?

The error estimtate gives

$$\frac{3(2^5)}{180n^4} \leq 0.000001 \quad \Rightarrow \quad n \geq \left(\frac{3(2^5)}{180(0.000001)}\right)^{1/4} = 27.02....$$

For $n = 28$ we have

$$S_{28} = 1.711259137.$$

The value of the integral that Maple gives is

$$\int_{-1}^{1} \exp(-x^2/2) \ dx = 1.711248783.$$