Math 113 Lecture #16
§7.8: Improper Integrals

Improper Integrals over Infinite Intervals. How do we make sense of and evaluate an *improper* integral like

$$\int_1^\infty \frac{1}{x^2}\,dx?$$

Can we apply the Fundamental Theorem of Calculus to evaluate this?

No, because the FTC only applies to integrals on finite intervals!

So instead we convert the integral over an infinite interval into a limit of integrals over finite intervals:

$$\lim_{A\to\infty} \int_1^A \frac{1}{x^2}\,dx.$$

Now we can apply the FTC to get

$$\lim_{A\to\infty} \int_1^A \frac{1}{x^2}\,dx = \lim_{A\to\infty} \left[-\frac{1}{x} \right]_1^A = \lim_{A\to\infty} \left[-\frac{1}{A} + 1 \right] = 1.$$

Since this limit exists, we say that the improper integral *converges*, and the value of this limit we take as the value of the improper integral:

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

Definition. If $\displaystyle\int_a^t f(x)\,dx$ exists for every $t \geq a$, then the improper integral

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty} \int_a^A f(x)\,dx$$

converges provided the limit exists (as a finite number); otherwise it diverges.

If $\displaystyle\int_t^b f(x)\,dx$ exists for every $t \leq b$, then the improper integral

$$\int_\infty^b f(x)\,dx = \lim_{B\to\infty} \int_B^b f(x)\,dx$$

converges provided the limit exists (as a finite number); otherwise it diverges.

If both improper integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_a^\infty f(x)\,dx + \int_{-\infty}^a f(x)\,dx.$$

The choice of a is arbitrary.

Example 1. For what values of $p > 0$ does the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge?

We have already seen that the improper integral converges when $p = 2$.

Now the case of $p = 1$ is different from $p \neq 1$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx = \lim_{A\to\infty} \big[\ln x\big]_1^A = \lim_{A\to\infty} \ln A = \infty.$$

So the improper integral diverges for $p = 1$.

Now for $p \neq 1$ we have

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{A\to\infty} \int_1^A \frac{1}{x^p}\,dx \\ &= \lim_{A\to\infty} \int_1^A x^{-p}\,dx \\ &= \lim_{A\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^A \\ &= \lim_{A\to\infty} \left[\frac{A^{-p+1}}{-p+1} - \frac{1}{-p+1}\right].\end{aligned}$$

If $p > 1$ then $-p + 1 < 0$ and the improper integral converges since the limit exists:

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

If $0 < p < 1$, then $-p + 1 > 0$ and so the limit does not exist, and the improper integral diverges.

The improper integral converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 2. For what values of $p > 0$ does the improper integral

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

converge?

We proceed using a limit:

$$\begin{aligned}\int_e^\infty \frac{1}{x(\ln x)^p}\,dx &= \lim_{A\to\infty} \int_e^A \frac{1}{x(\ln x)^p}\,dx \quad [u = \ln x,\ du = (1/x)dx] \\ &= \lim_{A\to\infty} \int_1^{\ln A} \frac{1}{u^p}\,du.\end{aligned}$$

But this is the problem in Example 1, and so we know that convergence occurs when $p > 1$ and divergence occurs when $0 < p \leq 1$.

Comparison Theorems for Improper Integrals. Often, it is not practical nor possible to directly evaluate an improper integral.

Instead we compare a particular improper integral with ones we know converge or diverge.

Theorem. Suppose that f and g are continuous with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

(a) If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is too.

(b) If $\displaystyle\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is too.

Example 3. Is $\displaystyle\int_1^\infty \frac{x}{1+x^7}\,dx$ convergent or divergent?

We could evaluate the integral directly by partial fractions, but is messy for this integral.

Instead, we use the Comparison Theorem.

Since $1 + x^7 \geq x^7$ for all $x \geq 1$, then

$$\frac{x}{1+x^7} \leq \frac{x}{x^7} = \frac{1}{x^6} \text{ for all } x \geq 1.$$

Now we know that

$$\int_1^\infty \frac{1}{x^6}\,dx$$

is a convergent improper integral, and so by the Comparison Theorem, the improper integral

$$\int_1^\infty \frac{x}{1+x^7}\,dx$$

is convergent too.

Example 4. Is $\displaystyle\int_e^\infty \frac{\ln x}{\sqrt{x}}\,dx$ convergent or divergent?

There is little hope of directly evaluate this improper integral, so we use the Comparison Theorem instead.

Since $\ln x \geq 1$ for $x \geq e$ and $x \geq \sqrt{x}$ for $x \geq e$, then

$$\frac{\ln x}{\sqrt{x}} \geq \frac{1}{x} \text{ for all } x \geq e.$$

Now we know that the improper integral

$$\int_e^\infty \frac{1}{x}\,dx$$

diverges, and so the improper integral

$$\int_e^\infty \frac{\ln x}{\sqrt{x}}\,dx$$

diverges as well

Improper Integrals with Infinite Discontinuous Integrands. What happens when the integrand has an infinite discontinuity?

This is another situation in which we have an improper integral, and as before, we use limits to decide if the improper integral converges or diverges.

Example 5. Is the improper integral $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$ convergent or divergent?

The infinite discontinuity occurs at the endpoint $x = 3$, and so we use the appropriate one-sided limit at this endpoint:

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{A\to 3^-}\int_2^A \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{A\to 3^-}\big[-2\sqrt{3-x}\big]_2^A \\ &= \lim_{A\to 3^-}\big[-2\sqrt{3-A}+2\big] \\ &= 2.\end{aligned}$$

So this improper integral converges.

Example 6. Does $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$ converge or diverge?

This improper integral has an infinite endpoint and an infinite discontinuity. So we split it up into two improper integrals:

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \lim_{A\to 0^+}\int_A^1 \frac{dx}{\sqrt{x}(1+x)} + \lim_{B\to\infty}\int_1^B \frac{dx}{\sqrt{x}(1+x)}.$$

Using the rational substitution $u^2 = x$ we can show that both improper integrals converge, and the the sum of these improper integrals is π.