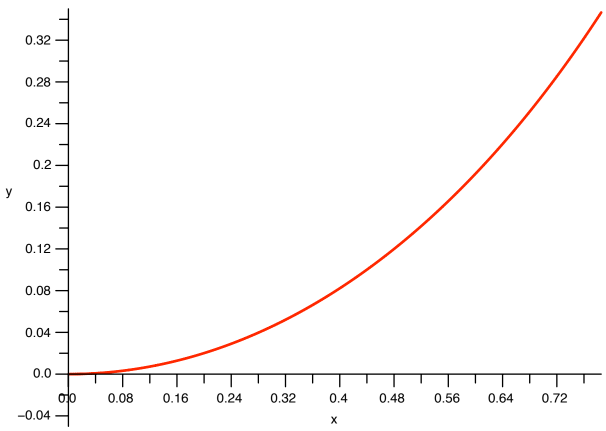
Math 113 Lecture #17
§8.1: Arc Length

Defining Arc Length. What do we mean by the "length" of a curve?

We know what is meant by the length of a straight line segment, and this is the key to defining the length of a curve.

Suppose that the curve is the graph of $y = f(x)$ where f is continuous on $[a, b]$.

Here is the graph of $y = \ln(\sec x)$ on $[0, \pi/4]$ for the purpose of illustration.



We can approximate the curve by a polygonal path, i.e., a path composed of straight-line segments.

We do this by subdividing the interval $[a, b]$ into subintervals of equal length $\Delta x = (b - a)/n$ for a positive integer n, and endpoints $x_i = a + i\Delta x$.

The points $P_i = (x_i, y_i)$ where $y_i = f(x_i)$ form the "corners" of the polygonal path and the straight lines segments between consecutive corners P_{i-1} and P_i connect the corners.

Can you sketch a picture of the approximating polygonal path to the curve?

The point of this polygonal path is that we known what is meant by the length of each of the straight-line parts in the polygonal path.

Let $|P_{i-1}P_i|$ denote the length of the straight line from P_{i-1} to P_i.

As $n \to \infty$ we expect that the length of the polygonal paths converges to the actual length of the curve:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

If this limit exists, we take its value *to be* the length of the curve.

Finding the length of a curve this way is impractical, so instead we try to formulate the limit L as an integral.

To do this we will have to ASSUME that f is differentiable, and that f' is continuous.

Now if we let $\Delta y = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

We are going to express the term Δy in terms of f'.

We CAN do this by the Mean Value Theorem because f is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i): there is x_i^* in (x_{i-1}, x_i) such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*).$$

In terms of Δx and Δy this is

$$\Delta y = f'(x_i^*)\Delta x.$$

The length of the line segment from P_{i-1} to P_i then becomes

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x)]^2} = \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x,$$

where we have assumed that $\Delta x > 0$.

Now the formula that defines the length of a curve $y = f(x)$ over $[a, b]$ is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

by the defintion of the definite integral.

The integral exists because assuming f' is continuous on $[a, b]$ implies that the integrand $\sqrt{1 + [f'(x)]^2}$ is continuous on $[a, b]$.

Computing Arc Length.

Generally, evaluating the definite integral for arc length is nearly impossible in an exact manner because the integrand is not nice.

Sometimes, an arc length has to be computed numerically.

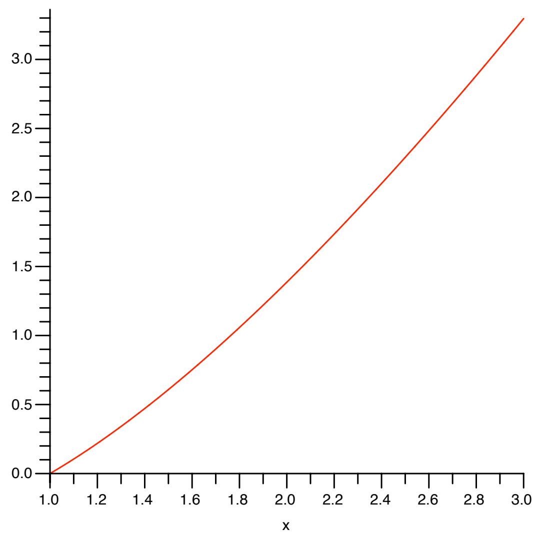
Example 1. Find the length of the curve $y = \ln(\sec x)$ over $[0, \pi/4]$. The graph of this curve is given on the previous page.

The derivative $\tan x$ exists and is continuous on $[0, \pi/4]$.

In this case we can compute the length of the curve exactly:

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\ dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx \\
&= \int_0^{\pi/4} \sec x\ dx \\
&= \Big[\ln(\sec x + \tan x) \Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right).
\end{aligned}$$

Example 2. Find the length of the curve $y = x \ln x$ over $[1, 3]$. Here is the graph of this curve.



The derivative $1 + \ln x$ exists and is continuous on $[1, 3]$, and so the the length of the curve is

$$L = \int_1^3 \sqrt{1 + [1 + \ln x]^2}\ dx = \int_1^3 \sqrt{2 + 2\ln x + (\ln x)^2}\ dx.$$

We approximate the value of this integral using Simpson's Rule:

$$S_{10} = 3.869618078, \quad S_{20} = 3.869617093, \quad S_{40} = 3.869616989.$$

Maple could not evaluate the integral exactly (i.e., it could not find an antiderivative), but for a numerical approximation it gave

$$L = 3.869616982.$$

The Arc Length Function.

The distance from the point $y = f(a)$ on a curve to any other point $y = f(x)$ for x in $[a, b]$ on the curve is given by the arc length function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\ dt.$$

As always for arc length, we are assuming that f is continuous on $[a, b]$ containing x, that f is differentiable on (a, b), and that f' is continuous on $[a, b]$.

By the Fundamental Theorem of Calculus, the derivative of the arc length function is

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Sometimes we see this derivative written in the Pythagorean form:

$$(ds)^2 = (dx)^2 + (dy)^2,$$

where dx is the run of a right angle triangle, dy is its rise, and ds is its hypothenuse.