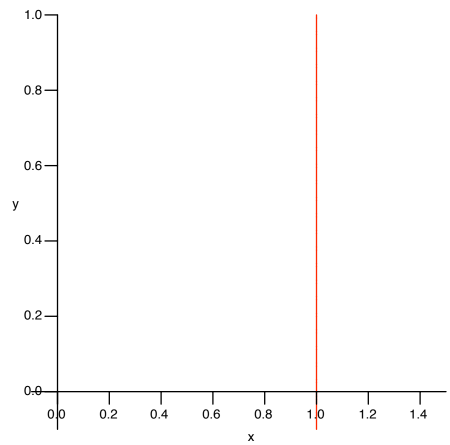
Math 113 Lecture #18
§8.2: Area of a Surface of Revolution

Defining the Area of a Surface of Revolution. Revolving a curve about a line forms a surface of revolution.

We can think of a surface of revolution as the lateral boundary of the solid of revolution obtained by revolution that curve about the line.

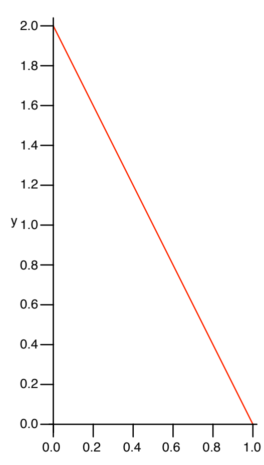
A simple example of a surface of revolution is the right cylinder of base radius r and lateral height h.



By cutting this right cylinder along the lateral side, we can unfold the right cylinder into a rectangle of base $2\pi r$ and height h, and thus is area is

$$A = 2\pi r h.$$

A less obvious example is that of a circular cone with base radius r and slant height l.



By cutting this circular cone along a vertical from the base to the point, we can unfold the circular cone into a sector of a circle with radius l and angle $\theta = 2\pi r/l$.

Remember the definition of angle: the length of the arc of the sector of the circle divided by the radius.

Recall that the area of a sector of radius l and angle θ is $(1/2)l^2\theta$.

Think of the area of a circle of radius l, which is πl^2, i.e., the angle θ is 2π.

Thus the area of the circular cone of base radius r and lateral height l is

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi r l = 2\pi\left(\frac{r-0}{2}\right)l.$$

Now for curves that are not already polygonal paths, we approximate the curve by a polygonal path and consider the straight-line segments of the polygonal path revolved around a line.

What we get when we do this are band or collars, each of which is a truncated circular cone, i.e., a frustum of a cone.

Each band has a slant height l and an upper and lower radii r_1 and r_2.

On the smaller radius side we place an imaginary circular cone that completes the band to full circular cone.

Let l_1 be the slant height of this imaginary circular cone, which for convenience is on the radius r_1 side.

The area of the band is now understood to be the difference of the areas of two circular cones, one of base radius r_1 and and slant height l_1 and the one with base radius r_2 and slant height $l + l_1$:

$$A = \pi r_2(l + l_1) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l].$$

The radii r_1 and r_2 and the slant heights l_1 and l are sides of two congruent right triangles, and so

$$\frac{l_1}{r_1} = \frac{l + l_1}{r_2}.$$

Multiplying this out gives

$$r_2 l_1 = r_1 l_1 + r_1 l \text{ or } (r_2 - r_1)l_1 = r_1 l.$$

We use this to eliminate the imaginary slant height l_1 from the formula for the area of the band:

$$A = \pi[(r_2 - r_1)l_1 + r_2 l] = \pi[r_1 l + r_2 l]$$

We now replace the two radii by their average $r = (r_1 + r_2)/2$ which gives the area of the band as

$$A = 2\pi r l.$$

We now apply this area formula for the band to the whole of the polygonal path that approximates the curve being revolved about a line.

Let $y = f(x)$ be the positive continuous function defined on $[a, b]$, and let the x-axis be the line of revolution.

We further assume that f' exists and is continuous on $[a, b]$ as well.

For a positive integer n, we divide $[a, b]$ into subintervals of equal length $\Delta x = (b - a)/n$ with endpoints $x_i = a + i\Delta x$.

Let $P_i = (x_i, y_i)$ be the point on the curve where $y_i = f(x_i)$.

The part of the surface revolved about the x-axis between x_{i-1} and x_i is approximated by the straight line segment $P_{i-1}P_i$ revolved about the x-axis.

The area of this band, i.e., frustum of a cone, is

$$A_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|,$$

where $(y_{i-1} + y_i)/2$ is the average radius, and $|P_{i-1}P_i|$ is the slant height.

Having assumed that f' exists and f' is continuous on $[a, b]$, we can write

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

for x_i^* in $[x_{i-1}, x_i]$, just as we did for arc length.

By the continuity of f when Δx is small we can write $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ and $y_i = f(x_i) \approx f(x_i^*)$, and so

$$\frac{y_{i-1} + y_i}{2} \approx f(x_i^*).$$

And so we now have the approximation of the area of the i^{th} band:

$$A_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

By taking the limit as $n \to \infty$, the sum of the approximations A_i approaches what we think of as the area of the surface of revolution:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx = \int_a^b 2\pi f(x)\ ds.$$

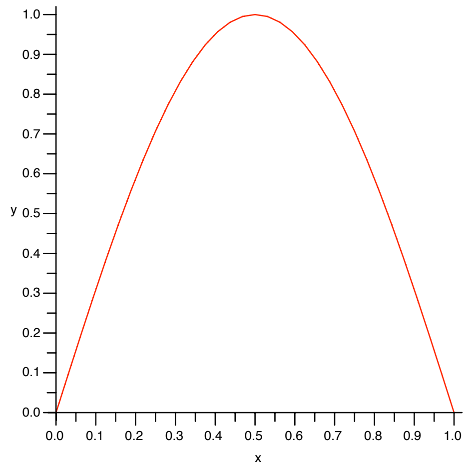
There is a similar formula for the area of a surface of revolution obtained by revolving a positive function $x = g(y)$ about the y-axis.

Computing Area of a Surface of Revolution.

Because the integral formula for the area of a surface of revolution involves arc length, computing surface area is not nice.

Sometimes we have to use an approximation like Simpson's Rule to estimate the surface area.

Example 1. Find the area of the surface obtained by revolving $y = \sin \pi x$, $0 \leq x \leq 1$, about the x-axis.



We compute the area of this surface of revolution using the formula:

$$
\begin{aligned}
S &= \int_0^1 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx \\
&= 2\pi \int_0^1 \sin \pi x \sqrt{1+[\pi\cos(\pi x)]^2}\,dx \quad [u = \pi\cos(\pi x),\ du = -\pi^2\sin(\pi x)] \\
&= -\frac{2}{\pi}\int_{\pi}^{-\pi} \sqrt{1+u^2}\,du \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi} \sqrt{1+u^2}\,du \quad [\text{Integration Table}] \\
&= \frac{2}{\pi}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})\right]_{-\pi}^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\pi}{2}\sqrt{1+\pi^2} + \frac{1}{2}\ln(\pi+\sqrt{1+\pi^2}) - \left(\frac{-\pi}{2}\sqrt{1+\pi^2} + \frac{1}{2}\ln(-\pi+\sqrt{1+\pi^2})\right)\right] \\
&= 2\sqrt{1+\pi^2} + \frac{1}{\pi}\ln\frac{\pi+\sqrt{1+\pi^2}}{-\pi+\sqrt{1+\pi^2}}.
\end{aligned}
$$