# Math 113 Lecture #19
## §8.3: Applications to Physics and Engineering

**Hydrostatic Force and Pressure.** If a thin plate is horizontal placed in a fluid, what is the force that the fluid above it exerts on the plate?

According to Newton's Second Law, it is the mass times the acceleration due to gravity.

Now if the plate has area $A$ meters squared and is at a depth of $d$ meters, and the fluid has a density of $\rho$ kilograms per cubic meter, then the volume and the mass of the fluid above the plate are

$$V = Ad \text{ and } m = \rho V = \rho A d.$$

If $g$ is the acceleration due to gravity, then applying Newton's Second Law gives

$$F = mg = \rho g A d.$$

The *pressure* on the plate from the fluid lying above it is the force per meter squared:

$$P = \frac{F}{A} = \rho g d.$$

The units of pressure are Newtons per meter squared, or Pascals (Pa).

This is a small unit, so generally kilopascals, kPA, are the units used in practice.

An important principle in fluid pressure is that at any point in the fluid, the pressure is the same in all directions.

Thus the pressure in *any* direction (not just from above) at a depth $d$ in a fluid with mass density $\rho$ is

$$P = \rho g d.$$

This permits defining the hydrostatic force on any object in a fluid, not just horizontally placed thin plates.

We will apply the pressure principle to vertically placed thin plates in fluilds.

Because the pressure changes as the depth changes, we have to take the divide and conquer approach to determine the force of the fluid on the plate.

Let $R$ be a planar region that represents the thin plate placed vertically in fluid.

Vertically place the $x$-axis with the origin at the top of the fluid, and the downward direction as the positive part of the $x$-axis.

In other words, $x$ is the depth.

Let $a$ be the depth of the top of the thin plate and $b$ the depth of its bottom.

Since the pressure changes with depth, we divide $[a, b]$ into equal subintervals of depth $\Delta x = (b - a)/n$ for a positive integer $n$, and endpoints $x_i = a + i\Delta x$.

On the horizontal part of the thin plate lying between $x_{i-1}$ and $x_i$, the pressure is near constant when $\Delta x$ is small.

If $P_i$ is this nearly constant pressure and $x_i^*$ is a sample point in $[x_{i-1}, x_i]$, then

$$P_i = \rho g x_i^*.$$

We need the area $A_i$ of the horizontal of the thin plate lying between $x_{i-1}$ and $x_i$ to get the force.

Suppose there is a continuous function $w(x)$ defined on $[a, b]$ for which $w(x)$ is the width of the thin plate at depth $x$.

Then an approximation of the area of the horizontal part of the plate lying between $x_{i-1}$ and $x_i$ is

$$A_i = w(x_i^*)\Delta x.$$

Thus an approximation of the force exerted by the liquid on the $i^{\text{th}}$ part of the vertical plate is

$$F_i = P_i A_i = \rho g x_i^* w(x_i^*)\Delta x.$$

Summing these approximations gives an approximation to the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho g x_i^* w(x_i^*)\Delta x.$$

Since we have assumed that $w$ is continuous on $[a, b]$, then taking the limit of these approximations of force gives the force on the plate as a definite integral:

$$F = \int_a^b \rho g x w(x)\ dx.$$

## Computing Hydrostatic Force.

The integrand for the force is nice enough that often we can compute the force exactly.

**Example 1.** A triangular shaped thin plate of base 4 m and height 5 m is placed in water with the 4 m base at the bottom and the tip at the surface of the water.

We need to determine the width function $w(x)$ for the plate, defined on $[0, 5]$.

At the depth $x$, the width $w(x)$ is the length of the horizontal line running from one side to the other side of the triangle.

Using similar triangles we find that

$$\frac{w(x)}{x} = \frac{4}{5} \text{ or } w(x) = \frac{4x}{5}.$$

Since $\rho = 1000$ kilograms per cubic meter, and $g = 9.8$ meters per second squared, the force exerted by the pressure of the water on the triangular plate is

$$\begin{aligned} F &= \int_0^5 \rho g x w(x)\ dx = \int_0^5 9800 \frac{4x^2}{5}\ dx \\ &= 7840 \int_0^5 x^2\ dx = 7840 \left[\frac{x^3}{3}\right]_0^5 \\ &= 7840 \left[\frac{125}{3}\right] = \frac{980000}{3} \text{ N.} \end{aligned}$$

**Example 2.** Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is at the bottom, with the end positioned vertically, in 10 feet of water.

We choose an $xy$-axis so that the origin is the center of the circular end with $y$ up as the positive $y$ direction, and the positive $x$ direction to the right.

We do this to take advantage of the circular geometry of the end of the cylindrical drum.

The equation for the circular end of the drum is $x^2 + y^2 = 9$.

The width of the circular end of the drum at depth $7 - y$ is

$$w(y) = 2\sqrt{9 - y^2}.$$

The area of a horizontal strip is

$$A(y) = w(y)dy = 2\sqrt{9 - y^2}dy.$$

Since the force density of water is $\delta = 62.5$ lb/ft$^3$, the pressure of the water on the horizontal strip at depth $7 - y$ is

$$P(y) = \delta(7 - y) = 62.5(7 - y).$$

Thus the force on the horizontal strip at depth $7 - y$ is

$$P(y)A(y) = 62.5(7 - y)2\sqrt{9 - y^2}dy.$$

The hydrostatic force on the circular end is

$$\begin{aligned} F &= \int_{-3}^{3} 125(7 - y)\sqrt{9 - y^2}\ dy \\ &= 125 \int_{-3}^{3} \{7\sqrt{9 - y^2} - y\sqrt{9 - y^2}\}\ dy \\ &= 125 \int_{-3}^{3} 7\sqrt{9 - y^2}\ dy - 125 \int_{-3}^{3} y\sqrt{9 - y^2}\ dy \\ &= 125 \int_{-3}^{3} 7\sqrt{9 - y^2}\ dy - 0 \\ &= 875\frac{9\pi}{2} \\ &= \frac{7875\pi}{2}. \end{aligned}$$

The zero value for the second integral occurs because the integrand is an odd function and the interval of integration is symmetric about the origin; the first integral is easy to evaluate because it is asking for half the area of a circle of radius 3.