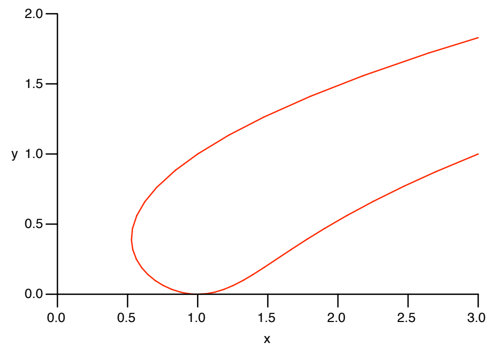
Math 113 Lecture #21
§10.1: Curves Defined by Parametric Equations

Parametric Equations and their Graphs. Imagine that a particle moves in the xy-plane along a curve C like the one shown below.



The curve C can not be the graph of a Cartesian equation $y = g(x)$ because C fails the Vertical Line Test.

Rather than the y coordinate being a function of the x coordinate on C (as is required by $y = g(x)$), maybe the x and y coordinates on C are functions of a third variable t:

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

These **parametric equations** are a new way of thinking about curves in the xy-plane.

The graph above is that of the parametric equations $x = t^4 - t + 1$, $y = t^2$ on an interval $[-1, b]$.

The initial point $(f(-1), g(-1)) = (3, 1)$ is where the curve C starts, and the terminal point $(f(b), g(b))$ is where the curve C ends.

As t varies from $t = -1$ to $t = b$, the point $(f(t), g(t))$ traces the curve C.

Example 1. Sketch the graph of the parametric equations

$$x = t^2 - 2t, \quad y = \sqrt{t}, \quad 0 \leq t \leq 3.$$

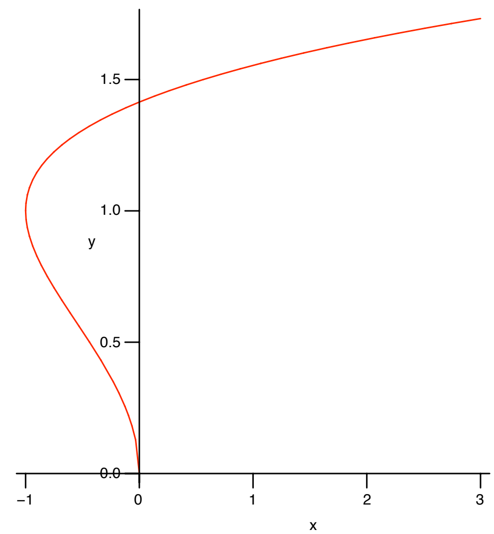
We can sketch the graph of these parametric equations by finding several points on the curve and connecting them in a continuous way.

The starting point is $(0, 0)$, and the ending point is $(3, \sqrt{3})$.

We compute two more points, say at $t = 1$ and $t = 2$.

These are the points $(-1, 1)$ and $(0, \sqrt{2})$.

Plotting these four points and connecting them in a continuous manner gives the graph below.



Cartesian and Parametric Equations. A Cartesian equation $y = g(x)$ can always be **parameterized**, that is written with x and y as functions of t, by setting

$$x = t, \text{ so that } y = g(x) = g(t).$$

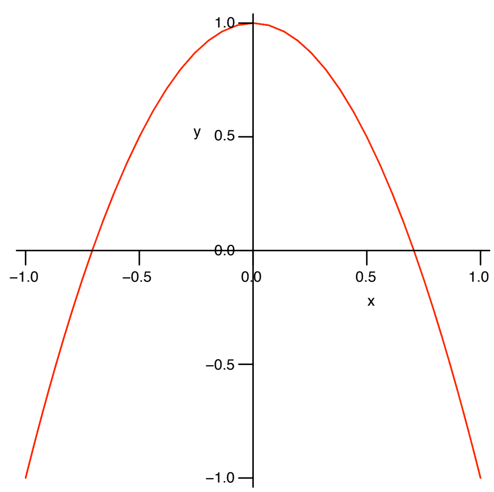
On the other hand, is it possible to *eliminate* the variable t from parametric equations $x = f(t), y = g(t)$ to get a Cartesian equation $y = h(x)$?

It is always possible when say $x = f(t)$ is invertible, so that $t = f^{-1}(x)$, and hence

$$y = g\big(f^{-1}(x)\big).$$

At other times the elimination of the parameter t requires some kind of an identity.

Example 2. Here is the graph of the parametric equations $x = \sin(t)$, $y = \cos(2t)$, for $0 \leq t \leq 2\pi$.



If this were the curve a moving particle traced, what is the motion like?

At initial point of the particle is $(0, 1)$, at the top of what appears to be a parabola.

Now as t goes positive, the value of $x = \sin t$ goes positive, so that particle moves to the right and down on the parabola-like curve.

What happens when t approaches $\pi/4$?

The value of $y = \cos(2t)$ approaches 0, at which point the particle crosses the x axis from the first quadrant into the fourth.

What happens as t approaches $\pi/2$? The value of $x = \sin t$ approaches 1 while the value of $y = \cos(2t)$ approaches -1.

This is the point at the tip of the right branch of the parabola-like curve.

What happens at t gets bigger than $\pi/2$? Where does the particle go now?

As t gets bigger than $\pi/2$, the value of $x = \sin t$ stays positive but decreasing from 1, while the value of $y = \cos(2t)$ increases from -1.

This means that the particle moves up the right branch of the parabola-like curve.

The particle is retracing its path!

What happens as t approaches $3\pi/4$? The value of $y = \cos(2t)$ approaches 0, so that the particle crosses the x-axis again, but this time from the fourth quadrant to the first.

What happens at t approaches π? Well, $x = \sin t$ approaches 0 and $y = \cos(2t)$ approaches 1, so that the particle approaches where it started.

Is there were the particle stops?

No, because as t gets bigger than π, the value of $x = \sin t$ turns negative, and now the particle is moving down the left branch of the parabola-like curve.

It will continue down this left side until at time $t = 3\pi/2$ it reaches the tip of the left branch, reverses direction, traveling up the left branch.

It returns to the starting point at time $t = 2\pi$, so that for the motion, the starting and terminal points are the same.

Now we kept saying parabola-like curve. It is really a parabola?

This is decided by eliminating the variable t. See any way to do this?

Neither x nor y is an invertible function of t, so that leaves us looking for an identity, presumably a trig identity.

What could it be? Well, you might remember that $\cos^2 t = \dfrac{1 + \cos 2t}{2}$.
Since $x^2 = \sin^2 t$, then

$$1 - x^2 = 1 - \sin^2 t = \cos^2 t = \frac{1 + \cos 2t}{2} = \frac{1 + y}{2}.$$

So indeed the curve of motion for the particle is a piece of a parabola.