Math 113 Lecture #22
§10.2: Calculus With Parametric Curves

Tangents. How do we determine the slope dy/dx and the concavity d^2y/dx^2 of a parametric curve $x = f(t)$, $y = g(t)$, $a \le t \le b$?

Well, we need to eliminate t somehow to get a Cartesian curve $y = F(x)$.

To do this we will make the simple supposition that $f'(t) \neq 0$ for all t in $[a, b]$.

Then f is monotonic (increasing if $f' > 0$ or decreasing if $f' < 0$) so that f is invertible and we can eliminate t.

Substitution of $x = f(t)$ and $y = g(t)$ into $y = F(x)$ gives

$$g(t) = y = F(x) = F\big(f(t)\big).$$

Here of course F is given by $F(x) = g\big(f^{-1}(x)\big)$, but we will not need this explicitly.

Differentiation of $g(t) = F\big(f(t)\big)$ with respect to t gives by the Chain Rule,

$$g'(t) = F'\big(f(t)\big)f'(t).$$

Since we assumed that $f'(t) \neq 0$ and since $x = f(t)$, we can rewrite this as

$$F'(x) = F'\big(f(t)\big) = \frac{g'(t)}{f'(t)}.$$

In Leibniz notation this is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

In other words, the slope of the parametric curve is the ratio of the derivative of the y part (the rise) over the derivative of the x part (the run).

The parametric curve has a horizontal tangent line whenever $dy/dt = 0$, and has a vertical tangent line whenever $dx/dt = 0$.

How about the d^2y/dx^2? How do we compute this for a parametric curve?

We go back to $g'(t) = F'\big(f(t)\big)f'(t)$ and differentiate this with respect to t using the product rule and the chain rule:

$$g''(t) = F''\big(f(t)\big)\big[f'(t)\big]^2 + F'\big(f(t)\big)f''(t).$$

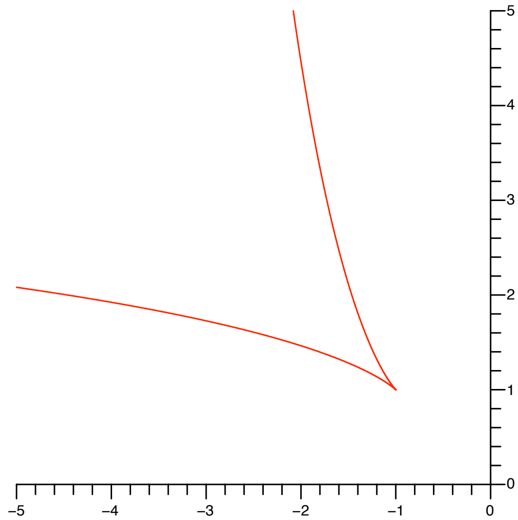
We solve this for $F''\big(f(t)\big) = F''(x)$ (since $x = f(t)$) which is d^2y/dx^2:

$$F''(x) = \frac{g''(t) - F'(x)f''(t)}{\big[f'(t)\big]^2}$$

We know already what $F'(x)$ is in terms of f and g, and so we put this in, and simplify to get

$$\frac{d^2y}{dx^2} = \frac{g''(t)f'(t) - g'(t)f''(t)}{\big[f'(t)\big]^3}.$$

Example 1. Find dy/dx and d^2y/dx^2 for $x = t - e^t$, $y = t + e^{-t}$, $-2 < t < 2$.



We compute the first derivative from the formula:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{1 - e^{-t}}{1 - e^t}.$$

At $t = 0$, this derivative is a $0/0$, which after applying L'Hospital's Rule, resolves to -1.

This parametric curve has a cusp at $t = 0$.

There are no horizontal or vertical tangents for this parametric curve.

We use the formula to compute the second derivative:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{g''(t)f'(t) - g'(t)f''(t)}{\big[f'(t)\big]^3} \\ &= \frac{e^{-t}(1 - e^t) + (1 - e^{-t})e^t}{(1 - e^t)^3} \\ &= \frac{e^{-t} - 1 + e^t - 1}{(1 - e^t)^3} \\ &= \frac{e^t + e^{-t} - 2}{(1 - e^t)^3}.\end{aligned}$$

The second derivative is not defined at $t = 0$ (use L'Hospital's Rule to see this).

But the second derivative is of one sign for $t < 0$ and of another sign for $t > 0$, so that the parametric curve switches concavity at $t = 0$.

Areas. Recall that when $F(x) \geq 0$, the area under the graph of a Cartesian curve $y = F(x)$ over $[a, b]$ when is

$$A = \int_a^b F(x)\, dx.$$

We apply this formula for area to a parametric curve $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ by replacing y with $g(t)$, replacing dx with $f'(t)\ dt$, where $a = f(\alpha)$ and $b = f(\beta)$:

$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t) f'(t)\ dt.$$

Example 2. Find the area enclosed by the parametric curve $x = t^2 - 2t$, $y = \sqrt{t}$, and the y-axis.

The initial point for this curve, you might recall, is the origin.

The curve crosses the y axis when $t = 2$, i.e., at the point $(0, \sqrt{2})$.

The area of this region is

$$\begin{aligned} A &= \int_0^2 g(t) f'(t)\ dt \\ &= \int_0^2 \sqrt{t}\ (2t - 2)\ dt \\ &= \int_0^2 \left(2t^{3/2} - 2t^{1/2}\right) dt \\ &= \left[\frac{4t^{5/2}}{5} - \frac{4t^{3/2}}{3}\right]_0^2 \\ &= \frac{4(2)^{5/2}}{5} - \frac{4(2)^{3/2}}{3}. \end{aligned}$$

Arc Length.

Recall that the arc length of a Cartesian curve $y = F(x)$ over $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

We will adapt this formula to a parametric curve $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ with the assumption that $dx/dt = f'(t) > 0$.

The parametric curve is traveled once from left to right at t increases from α to β, where $a = f(\alpha)$ and $b = f(\beta)$.

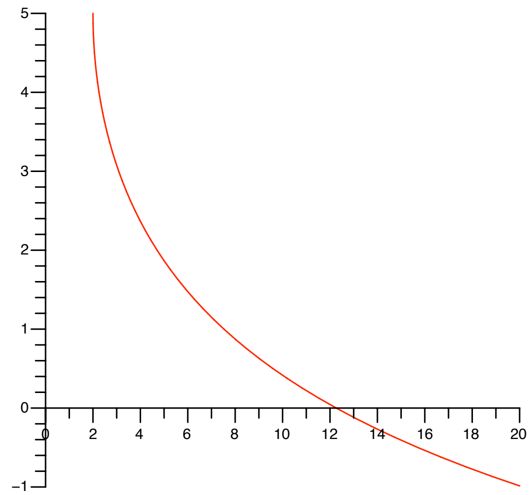
Applying the Substitution Rule gives

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{1 + \left(\frac{g'(t)}{f'(t)}\right)^2}\ f'(t)\ dt \\ &= \int_\alpha^\beta \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\ dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \end{aligned}$$

A divide and conquer argument will show that this formula for the arc length of a parametric curve holds whether or not $f'(t) > 0$.

This is assuming that f' and g' are continuous on $[\alpha, \beta]$, and that the parametric curve is traveled exactly once.

Example 3. Find the arc length of the parametric curve $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$.



We apply the arc length formula:

$$\begin{aligned} L &= \int_0^3 \sqrt{(e^t - e^{-t})^2 + 4}\ dt \\ &= \int_0^3 \sqrt{4\left(\frac{e^t - e^{-t}}{2}\right)^2 + 4}\ dt \\ &= 2\int_0^3 \sqrt{\sinh^2 t + 1}\ dt \\ &= 2\int_0^3 \cosh t\ dt \\ &= 2\Big[\sinh t\Big]_0^3 \\ &= 2\sinh(3). \end{aligned}$$

Surface Area. Suppose for a parametric curve $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, that f' and g' are continuous on $[\alpha, \beta]$, and that $g(t) \ge 0$.

Then the area of the surface of revolution obtained by revolving the parametric curve about the x-axis is

$$SA = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt.$$