# Math 113 Lecture #23
# §10.3: Polar Coordinates

**Conversion Between Cartesian and Polar Coordinates.** The Cartesian coordinates $(x, y)$ represent the plane.

The **polar coordinates** $(r, \theta)$ are a different way to represent the same plane.

To define polar coordinates, we start by identifying a **pole** or origin, labeled $O$, usually taken to be the same as the origin in Cartesian coordinates.

Next we draw a half-ray starting at $O$ called the **polar axis**, which is usually draw horizontal to coincide with the positive $x$-axis in Cartesian coordinates.

For any point $P$ in the plane, we set $r$ to be the distance the point $P$ is from $O$:

$$r = |OP|.$$

We set $\theta$ to be "the" angle that the line segment $OP$ makes with the polar axis.

We measure $\theta$ in radians, with positive angle $\theta$ taken in the counterclockwise direction, and negative $\theta$ taken in the clockwise direction.

Notice that the pole $O$ is represented by $r = 0$ for any angle $\theta$.

For $P \neq O$, the angle $\theta$ is not unique, but is determined up to an integer multiple of $2\pi$ radians.

The way we set $r$ meant it is not negative, but we can easily extend $r$ to the negative case.

The point $(-r, \theta)$ corresponds to the point on the ray of angle $\theta$ with the polar axis, but in the opposite direction to that of $(r, \theta)$.

Or in other words, the polar coordinates $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point in the plane.

What is the relationship between polar coordinates $(r, \theta)$ and Cartesian coordinates $(x, y)$?

Think of how $r$ and $\theta$ are defined for a point $P = (x, y)$: there is a right-angled triangle whose hypothenuse is the line segment $OP$ which has length $r$, whose side adjacent to the angle $\theta$ is $x$, and whose side opposite the angle $\theta$ is $y$.

By the definitions of sine and cosine we therefore have

$$x = r\cos\theta, \quad y = r\sin\theta.$$

This tells us how to convert from polar coordinates to Cartesian coordinates.

How do we go from Cartesian coordinates to polar coordinates?

We make use of the most well known trigonometry identity:

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

This gives us $r$ in terms of $x$ and $y$, which is nothing more than the Pythagorean Theorem.

How about the angle $\theta$? Well,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

Now you might be tempted to apply arctan to both sides of this, but remember that arctan gives an angle between $-\pi/2$ and $\pi/2$, which comprises the first and fourth quadrant.

How about the second and third quadrants? Well, that depends on the point $(x, y)$, specifically the $x$ part.

If $x < 0$, then the point $(x, y)$ is in the second or third quadrant and we have

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi.$$

Otherwise, if $x > 0$ we are in the first or fourth quadrant, and so here,

$$\theta = \arctan\left(\frac{y}{x}\right).$$

**Example 1.** Find polar coordinates for the points whose Cartesian coordinates are $(3\sqrt{3}, 3)$ and $(1, -2)$.

For the first point we compute

$$r = \sqrt{x^2 + y^2} = \sqrt{27 + 9} = \sqrt{36} = 6,$$

and since $x > 0$, we compute

$$\theta = \arctan\left(\frac{3}{3\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

[Recall that $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$ so that $\tan(\pi/6) = 1/\sqrt{3}$.]

For the second point, we compute

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 4} = \sqrt{5},$$

and since $x > 0$,

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan(-2).$$

## Polar Curves.

A **polar equation** is of the form $F(r, \theta) = 0$.

The graph of a polar equation is the set of points $P$ for which at least one representative $(r, \theta)$ satisfies $F(r, \theta) = 0$.

Another name for the graph of a polar equation is **polar curve**.

The polar curve $r = 2$ is the circle centered at the origin with radius 2, while the polar curve $\theta = \pi/4$ is the line passing through the origin with slope 1.

Most polar equations we will see can be written in the form $r = f(\theta)$ or $\theta = g(r)$.

**Example 2.** Sketch the graph of the polar curve $r = 4\sin 3\theta$, $0 \le \theta < \pi$.

We start with the initial point corresponding to $\theta = 0$, which point is the origin.

We consider what happens to $r$ as $\theta$ increases away from 0: the value of $4\sin(3\theta)$ increases towards 4 which is reached at the angle $\theta = (\pi/2)/3 = \pi/6$.

For $\theta$ increasing from $\pi/6$, the value of $r = 4\sin(3\theta)$ now decreases to 0 which is achieved when $\theta = \pi/3$.

So the polar curve passes through that origin when $\theta$ passes through $\pi/3$.

For $\theta$ increasing from $\pi/3$, the values of $r = 4\sin 3\theta$ are negative, and so the polar curves goes into the third quadrant.

As $\theta$ approaches $\pi/2$ (i.e., as $3\theta$ approaches $3\pi/2$), the negative values of $r = 4\sin 3\theta$ approach a minimum value of $-4$.

Thus the point $(-4, \pi/2)$ is on the polar curve.

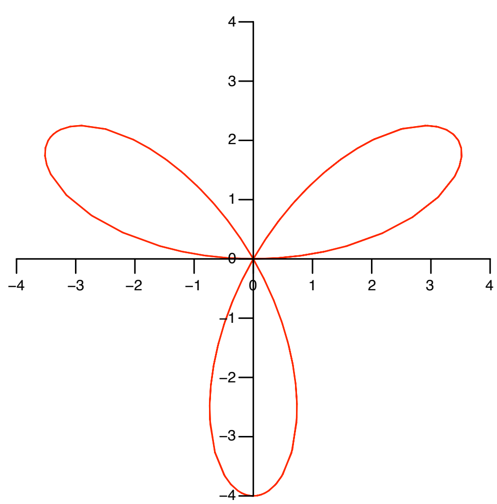
Beyond $\theta = \pi/2$, the values of $r = 4\sin 3\theta$ are still negative but increase towards 0, achieve 0 when $\theta = 2\pi/3$.

These points on the polar curve lie in the fourth quadrant, since $r$ is negative and $\theta$ is in the second quadrant.

Continuing $\theta$ past $2\pi/3$, the values of $r = 4\sin 3\theta$ become positive, and the points on the polar curve are in the second quadrant.

A maximum value of $r = 4$ is achieved when $\theta = 5\pi/6$, after which the value of $r$ decrease towards 0 which is reached as $\theta$ approaches $\pi$.

The polar curve we have is a three petal flower.




We can use polar equations to describe regions in the plane.

**Example 3.** What is the region describe by $1 \leq r \leq 2$ and $\pi/4 \leq \theta \leq 3\pi/4$?

Can you sketch this region? Can you visualize it?

This region is called a **polar rectangle**.

We can also convert Cartesian equations into polar equations using $x = r\cos\theta$ and $y = r\sin\theta$.

**Example 4.** Find a polar equation for $xy = 4$.

The polar equation for this hyperbola is

$$4 = (r\cos\theta)(r\sin\theta) = r^2\cos\theta\sin\theta.$$

Solving this for $r^2$ gives

$$r^2 = \frac{4}{\cos\theta\sin\theta}.$$

## Tangents for Polar Curves.

For a polar curve of the form $r = f(\theta)$, we can compute tangents (and everything else we learned for parametric curves) because we can use $\theta$ as the parameter:

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

The slope for such a parametric curves is

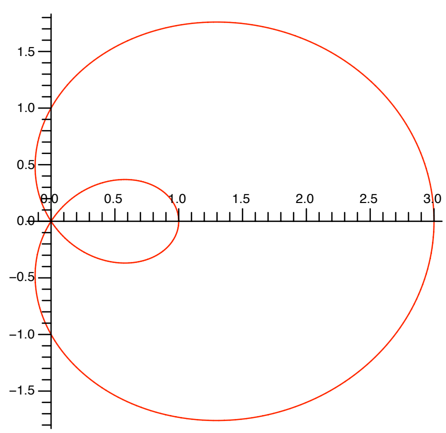
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

**Example 5.** Find the slope of the polar curve $r = 1 + 2\cos\theta$.

Applying the derivative formula, we have

$$\frac{dy}{dx} = \frac{-2\sin^2\theta + (1 + 2\cos\theta)\cos\theta}{-2\sin\theta\cos\theta - (1 + 2\cos\theta)\sin\theta}.$$

Here is a graph of this polar curve.




There are four points where this polar curve has horizontal tangent lines? Can you find them?