Math 113 Lecture #24
§10.4: Areas and Lengths in Polar Coordinates

Recall for a Cartesian curve $y = F(x)$ on $a \leq x \leq b$ with F nonnegative, that the area under the graph of F is

$$A = \int_a^b F(x)\,dx,$$

and that the arc length of the graph of $y = F(x)$ is

$$L = \int_a^b \sqrt{1 + [F'(x)]^2}\,dx.$$

The form of these formulas for area and arc length depended on the Cartesian coordinate system in which the curves were rendered.

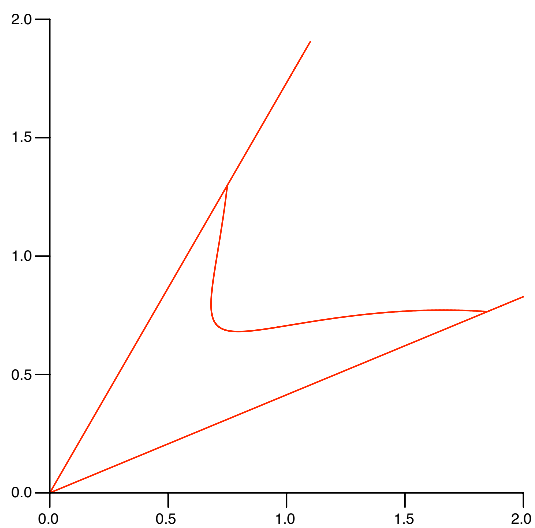
What might these formulas look like in polar coordinates? We are going to derive the form of these formulas in the polar coordinate system.

Area in Polar Coordinates. We begin by recalling the formula of the area of a sector of a circle of radius r and angle (in radians) θ:

$$A = \frac{r^2\theta}{2}.$$

However, not every polar curve is a circle.

How do we find the area $\mathcal{R}$ bounded by the rays $\theta = a$ and $\theta = b$ with $0 \leq a < b < 2\pi$, and the polar curve $r = f(\theta)$ for f positive and continuous? Here is a picture of such a region.



We subdivide the angle interval $[a, b]$ into equal subintervals of equal length $\Delta\theta = (b-a)/n$ and endpoints $\theta_i = a + i\Delta\theta$.

Choose sample points θ_i^* in $[\theta_{i-1}, \theta_i]$.

The piece $\mathcal{R}_i$ of $\mathcal{R}$ that lies between the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$ looks like a sector of a circle for $\Delta\theta$ small enough.

This suggests we can estimate the area of $\mathcal{R}_i$ by a sector of a circle with radius $f(\theta_i^*)$ and angle $\Delta\theta$:

$$A_i \approx \frac{[f(\theta_i^*)]^2 \Delta\theta}{2}.$$

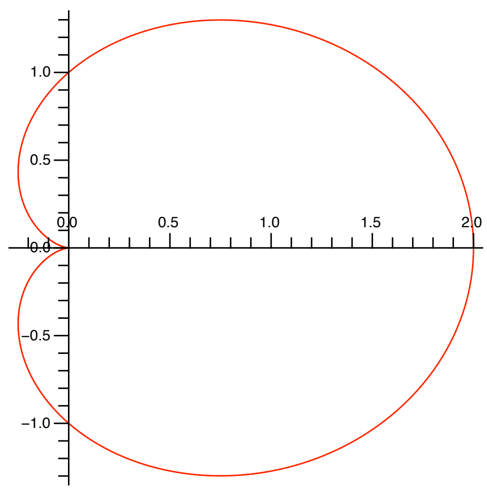
Summing these approximations gives an approximation for the area of $\mathcal{R}$:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{[f(\theta_i^*)]^2 \Delta\theta}{2}.$$

Since we are assuming that f is continuous on $[a, b]$, then the limit of the approximations as $n \to \infty$ is a definite integral that gives the area of $\mathcal{R}$:

$$A = \int_a^b \frac{[f(\theta)]^2}{2}\, d\theta = \int_a^b \frac{r^2}{2}\, d\theta.$$

Example 1. Find the area enclosed by the top half of the cardioid $r = 1 + \cos\theta$.

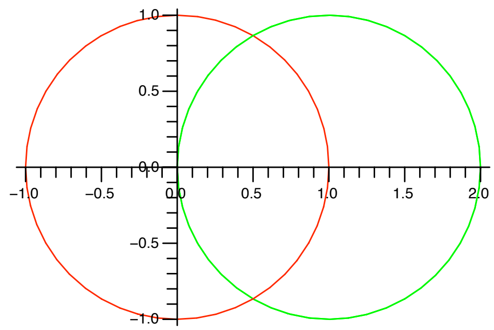


The top half of this cardioid starts at the point $(2, 0)$, i.e., $\theta = 0$, and ends at $(0, 0)$, i.e., $\theta = \pi$.

The area is

$$\begin{aligned} A &= \int_0^\pi \frac{[1+\cos\theta]^2}{2}\, d\theta = \frac{1}{2}\int_0^\pi \left(1 + 2\cos\theta + \cos^2\theta\right)\, d\theta \\ &= \frac{1}{2}\int_0^\pi \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)\, d\theta = \frac{1}{2}\int_0^\pi \left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)\, d\theta \\ &= \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^\pi = \frac{1}{2}\left[\frac{3\pi}{2}\right] = \frac{3\pi}{4}. \end{aligned}$$

Example 2. Find the area that lies outside $r = 1$ and inside $r = 2\cos\theta$.



We need to find the angles of intersection:

$$1 = 2\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{3}.$$

Following the principle of how to compute the area between curves, the area here is between the "outer" curve and the "inner" curve:

$$\begin{aligned} A &= \int_a^b \frac{1}{2}\big([f(\theta)]^2 - [g(\theta)]^2\big)\ d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\big(4\cos^2\theta - 1\big)\ d\theta \\ &= \int_0^{\pi/3} \big(4\cos^2\theta - 1\big)\ d\theta \quad \text{[use symmetry]} \\ &= [2\theta + \sin 2\theta - \theta]_0^{\pi/3} \\ &= \frac{\pi}{3} + \frac{\sqrt{3}}{2}. \end{aligned}$$

Arc Length in Polar Coordinates.

We do not need to divide and conquer to get the arc length formula in polar coordinates, but can adapt the formula in Cartesian coordinates for a parameterized curve:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Using $x = r\cos\theta$ and $y = r\sin\theta$, we can parameterize the polar curve $r = f(\theta)$, $a \le \theta \le b$ (thinking of t as θ):

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

We compute the derivatives:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

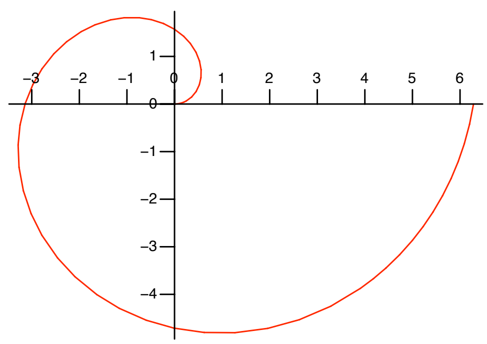
Then using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2.\end{aligned}$$

So the arc length of a polar curve is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 3. Find the arc length of $r = \theta$ on $[0, 2\pi]$.



The arc length of this polar curve is

$$L = \int_0^{2\pi} \sqrt{\theta^2 + 1}\; d\theta.$$

We use the integral formula #21 from the Reference Pages:

$$L = \left[\frac{\theta\sqrt{\theta^2+1}}{2} + \frac{\ln(\theta + \sqrt{\theta^2+1})}{2}\right]_0^{2\pi} = \frac{2\pi\sqrt{4\pi^2+1}}{2} + \frac{\ln(2\pi + \sqrt{4\pi^2+1})}{2}.$$