Math 113 Lecture #25 §11.1: Sequences

You have been using sequences for some time. Where? With Riemann sums.

For a continuous f and each positive integer n, the Riemann sum

$$a_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

gives an approximation of the definite integral

$$\int_{a}^{b} f(x) dx.$$

The sequence of numbers a_1 , a_2 , a_3 , etc. approaches or converges to the value of the definite integral as $n \to \infty$.

We will explore in Chapter 11 this fundamental notion of a convergent sequence of numbers.

This idea of approximating an exact number through a convergent sequence of numbers is profound.

To illustrate, the irrational number π is the limit of the sequence of rational numbers 3, 3.1, 3.14, etc.

But how do we get the exact value of π and not just an approximation?

We get the exact value of π through a sequence of numbers that converges to it.

Convergence and Divergence of Sequences. A sequence is a list of numbers given in a definite order:

 $a_1, a_2, \ldots, a_n, \ldots$

Alternatively, a sequence is a function from the positive integers to the reals, i.e., for each positive integer n there is associated a number a_n .

We say that a_1 is the first term of the sequence, a_2 the second term, and a_n is the n^{th} term.

Notational we write a sequence as

$$\{a_n\}$$
 or $\{a_n\}_{n=1}^{\infty}$.

Sometimes there is a simple function defining the terms of a sequence such as

$$a_n = \frac{(-1)^n \ln(n+1)}{n!}$$

At other times there is no simple function defining the terms of a sequence such as

$$2, 7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots$$

Do you recognize the digits in this sequence?

The notion of convergence of a sequence is about where the terms in the sequence are going.

For example, the terms in sequence $a_n = n/(n+1)$ are approaching 1 as $n \to \infty$.

We have a precise way of encoding what we mean by "the terms of the sequence approach a number."

Definitions. A sequence $\{a_n\}$ has a **limit** L and we write

$$\lim_{n \to \infty} a_n = L, \text{ or } a_n \to L \text{ as } n \to \infty,$$

if for every real $\epsilon > 0$ there is a corresponding positive integer N such that

$$|a_n - L| < \epsilon$$
 for all $n \ge N$.

In other words, no matter how close we focus in on L (i.e., the choice of ϵ), the terms in the sequence eventually come into and stay in our view (i.e., a_n for all $n \ge N$).

We say the sequence $\{a_n\}$ converges (or is convergent) if it has a limit, and otherwise say it **diverges** (or is divergent).

If the terms of a sequence are defined by a "nice" function, then there is a simple way to check for convergence.

Theorem. Suppose f is defined on $[1, \infty)$. If $a_n = f(n)$ for all $n \ge 1$, and

$$\lim_{x \to \infty} f(x) = L,$$

then $\{a_n\}$ converges, and $\lim_{n\to\infty} a_n = L$.

Now there are lots of way a sequence can diverge. Here is one way.

Definition. The symbolic phrase

$$\lim_{n \to \infty} a_n = \infty$$

means that for every positive integer M there is an integer N such that $a_n \ge M$ whenever $n \le N$.

There is a similar understanding of what its means for $\lim_{n\to\infty} a_n = -\infty$.

Example 1. The sequence $a_n = e^{-1/n}$ converges to 1 as $n \to \infty$, while the sequence $b_n = \ln n$ diverges to ∞ as $n \to \infty$.

Limit Laws for Convergent Sequences. When it comes to finding the limit of a convergent sequence, it is sometimes easier to split the sequence into pieces that converge, and find the limits of these convergent pieces to compute the limit of the original convergent sequence.

Here is a list of Limit Laws that help in doing this. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and c and p are constants with p > 0, then

$$\lim_{n \to \infty} (ca_n \pm b_n) = c \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$
$$\lim_{n \to \infty} a_n b_n = \left(\lim_{n \to \infty} a_n\right) \left(\lim_{n \to \infty} b_n\right),$$
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \text{ if } \lim_{n \to \infty} b_n \neq 0,$$
$$\lim_{n \to \infty} (a_n)^p = \left(\lim_{n \to \infty} a_n\right)^p \text{ if } a_n \ge 0.$$

Example 2. Here is an illustration of the Limit Laws:

$$\lim_{n \to \infty} \sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\lim_{n \to \infty} \frac{3+2n^2}{8n^2+n}} = \sqrt{\lim_{n \to \infty} \frac{3/n^2+2}{8+1/n}} = \sqrt{\frac{\lim_{n \to \infty} 3/n^2+2}{8+\lim_{n \to \infty} 1/n}} = \sqrt{\frac{2}{8}} = \frac{1}{2}.$$

Theorems about Convergence Sequences. Sometimes the limit of a convergence sequence is not easy to find, but it can be squeezed in between two convergent sequences whose limits are easy to find.

Squeeze Theorem. If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n \to \infty} b_n = L$$

Another useful theorem about convergent sequences is the following.

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose $\lim_{n\to\infty} |a_n| = 0$. Then by a Limit Law,

$$\lim_{n \to \infty} \left(-|a_n| \right) = 0.$$

Observe that $-|a_n| \leq a_n \leq |a_n|$ for all n. Since the limit of the outer two sequences are both 0, the Squeeze Theorem gives the result.

Another useful theorem for convergence sequences extends the last of the Limit Laws to any continuous function. **Theorem.** If $\lim_{n\to\infty} a_n = L$ and a function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L).$$

Example 3. Find the limit of

$$a_n = \cos\left(\frac{n!}{n^n}\right).$$

By the last Theorem we first investigate the sequence

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right) \le \frac{1}{n}.$$

Since $0 \le n!/n^n \le 1/n$, and the limit of the first and last terms here are 0, the Squeeze Theorem gives

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0.$$

The limit of $\{a_n\}$ is therefore 1.

Boundedness and Monotonicity of Sequences. A sequence $\{a_n\}$ is increasing if

$$a_n < a_{n+1}$$
 for all $n \ge 1$,

is decreasing if

 $a_n > a_{n+1}$ for all $n \ge 1$,

is **bounded above** if there is a number M such that

 $a_n \leq M$ for all $n \geq 1$,

and is **bounded below** if there is a number m such that

$$m \leq a_n$$
 for all $n \geq 1$

A increasing or decreasing sequence is also called a **monotonic** sequence.

A sequence that is bounded above and bounded below is called **bounded**.

Monotonic Sequence Theorem. Every bounded, monotonic sequence converges.

This Theorem is a consequence of the Completeness Axiom for the real numbers \mathbb{R} which expresses the belief that there are no holes or gaps in the real numbers.

An *upper bound* for a nonempty set S of real numbers is a real number M such that

$$s \leq M$$
 for all $s \in S$.

A least upper bound of S is a real number L such that L is an upper bound for S and for any upper bound M of S we have $L \leq M$.

The **Completeness Axiom** states that every nonempty set of real numbers S that has an upper bound has a least upper bound.

Much of Calculus is a consequence of this fundamental *belief* for the real numbers.