Math 113 Lecture #26
§11.2: Series

Partial Sums and Series.

It is easy to add a finite list of numbers.

But what about adding an infinite list of numbers together?

Consider the list of infinitely many numbers determined by a sequence $\{a_i\}$.

Adding this infinite list of numbers gives an **infinite series** (or series for short),

$$a_1 + a_2 + a_3 + \cdots,$$

which we denote by

$$\sum_{i=1}^{\infty} a_i \text{ or } \sum a_i.$$

What do we mean by adding an infinite list of numbers together?

We approach this question by using what we know, and that is that we know how to add a finite list of numbers.

The n^{th}-**partial sum** of a series is the addition of a finite list of numbers from the sequence, namely

$$s_n = \sum_{i=1}^{n} a_i.$$

Example 1. What is the n^{th}-partial sum of

$$\sum_{i=1}^{\infty} i^2?$$

Well it is

$$s_n = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

The sequence of partial sums $\{s_n\}$ diverges to ∞ as $n \to \infty$.

Example 2. What is the n^{th}-partial sum of

$$\sum_{i=1}^{n} \frac{1}{2^i}?$$

It is

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

The sequence of partial sums converges to 1 as $n \to \infty$.

Definition. A series $\sum_{i=1}^{\infty} a_i$ is called **convergent** if its sequence of partial sums s_n is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n.$$

Otherwise the series is called **divergent**.

Geometric Series.

A common series used is the geometric series

$$a + ar + ar^2 + \cdots + ar^{i-1} + \cdots = \sum_{i=1}^{\infty} ar^{i-1},$$

where a is nonzero constant, and r a constant.

Let us see if we can determine when the geometric series is convergent (and what it converges to) or divergent.

The way we make this determination is through the partial sums:

$$s_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiplying this partial sum through by r gives something that looks like the next partial sum:

$$rs_n = r\sum_{i=1}^{n} ar^{i-1} = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Combining these two finite sums by subtraction gives

$$s_n - rs_n = a - ar^n.$$

Solving for the partial sum when $r \neq 1$ gives

$$s_n = \frac{a(1-r^n)}{1-r}.$$

The term $1 - r^n \to 1$ as $n \to \infty$ when $-1 < r < 1$, which means that the partial sums converge:

$$\sum_{i=1}^{\infty} ar^{i-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

For $r = 1$, the geometric series diverges because $a \neq 0$:

$$s_n = \sum_{i=1}^{n} a1^{i-1} = a + a + \cdots + a = na \to \pm\infty.$$

For $r = -1$, the geometric series also diverges:

$$s_n = \sum_{i=1}^{n} a(-1)^{i-1} = \begin{cases} 0 & n \text{ even}, \\ a & n \text{ odd}. \end{cases}$$

Finally for $|r| > 1$, the geometric series diverges because it partial sums diverge.

Example 3. If convergent, find the sum of the series

$$\sum_{i=1}^{\infty} 6(0.9)^{i-1}.$$

We recognize this as a geometric series with $a = 6$ and $r = 0.9$, and so it is convergent and converges to

$$\frac{a}{1-r} = \frac{6}{1-0.9} = \frac{6}{0.1} = 60.$$

Divergence Series.

What do the terms a_n in a convergent series $\sum a_n$ do?

Once we know what this *necessary condition* of convergence is, we can use it to detect divergence.

Theorem. If the series $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. We know that the n^{th}-partial sum of the convergent series is

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n.$$

The difference of two consecutive partial sums is

$$s_n - s_{n-1} = a_n.$$

Since we are assuming that the series converges, we are also assuming that the sequence of partial sums converges:

$$s = \lim_{n\to\infty} s_n.$$

On the other hand, we have the other sequence s_{n-1} of partial sums which converges to the same thing:

$$s = \lim_{n\to\infty} s_{n-1}.$$

We now use the Limit Law for a difference of convergent sequences:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \big(s_n - s_{n-1}\big) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus we have shown that the terms a_n in the series go to zero as $n \to \infty$. $\square$

The *contrapositive* of this Theorem gives a test for divergence of a series.

Theorem. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Example 4. The series $\sum_{i=1}^{\infty} 2^{1/n}$ diverges because

$$\lim_{n\to\infty} 2^{1/n} = 1 \neq 0.$$

The *converse* of the first Theorem in this subsection is false: knowing that $a_n \to 0$ as $n \to \infty$ does NOT guarantee that $\sum a_n$ converges.

An example of this is the Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{i}.$$

We consider the partial sums s_2, s_4, s_8, etc., those of the form s_{2^n}.

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2}, \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\
&= 1 + \frac{1}{2} + \frac{1}{2} \\
&= 1 + \frac{2}{2}, \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= 1 + \frac{3}{2}.
\end{aligned}$$

We see a pattern here, namely that

$$s_{2^n} > 1 + \frac{n}{2}.$$

This says that the sequence of partial sums has terms in it that go to infinity, and so

$$\lim_{n\to\infty} s_n \text{ does not exist.}$$

Thus the Harmonic series diverges.

We will also show by an improper integral that the Harmonic series diverges.

Algebraic Rules for Convergence Series.

Convergent series enjoy some of the usual algebraic operations.

For convergent series $\sum a_n$ and $\sum b_n$ and a constant c, we have that

$$\sum \left(ca_n \pm b_n\right) = c\sum a_n \pm \sum b_n.$$

Example 5. If convergent, find the value of

$$\sum_{i=1}^{\infty} \left(\frac{3}{i(i+3)} + \frac{1}{7^{i-1}} \right).$$

We consider each part of the series separately for convergence.

By partial fractions, the partial sums for the first part are

$$\begin{aligned}
s_n &= \sum_{i=1}^{n} \frac{3}{i(i+3)} \\
&= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+3}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{4} - \frac{1}{5} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right) \\
&\to \left(1 + \frac{1}{2} + \frac{1}{3}\right) \\
&= \frac{6+3+2}{6} \\
&= \frac{11}{6}.
\end{aligned}$$

We used a *telescoping* sum technique to get the value of the sum.

The second part is a geometric series with $a = 1$ and $r = 1/7$, and so it converges too:

$$\sum_{i=1}^{\infty} \frac{1}{7^{-1}} = \frac{1}{1 - 1/7} = \frac{7}{6}.$$

Putting it all together gives the sum of

$$\sum_{i=1}^{\infty} \left(\frac{3}{i(i+3)} + \frac{1}{7^{i-1}}\right) = \frac{11}{6} + \frac{7}{6} = 3.$$

Example 6. A medication is administered to a patient at the same time every day. Suppose C_n is the concentration (in mg/mL) of the medication in the patient's bloodstream after the injection on the n^{th} day. Before the injection the next day, only 40% of the medication remains in the patient's bloodstream, and the daily injection raises the concentration by 0.3 mg/mL.

We assume that $C_0 = 0$, i.e., that before the first injection there in none of the medication in the bloodstream.

On the $(n+1)^{\text{th}}$ day, the concentration of the medication just before the injection is $0.4C_n$, and after the injection it is

$$C_{n+1} = 0.3 + 0.4C_n.$$

With this we calculate

$$\begin{aligned}
C_1 &= 0.3 + 0.4(0) = 0.3, \\
C_2 &= 0.3 + 0.4(0.3) = 0.42, \\
C_3 &= 0.3 + 0.4(0.42) = 0.468.
\end{aligned}$$

Does the sequence (C_n) converge as $n \to \infty$? If so, to what?

And is (C_n) the sequence of partial sums for a series?

To see what is happening we reconsider the first few terms of (C_n):

$$\begin{aligned}
C_1 &= 0.3 + 0.4(0) = 0.3,\\
C_2 &= 0.3 + 0.4C_1 = 0.3 + 0.4(0.3) = 0.3 + 0.3(0.4),\\
C_3 &= 0.3 + 0.4C_2 = 0.3 + 0.4(0.3 + 0.3(0.4)) = 0.3 + 0.3(0.4) + 0.3(0.4)^2.
\end{aligned}$$

The pattern that appears here suggests that

$$C_n = 0.3 + 0.3(0.4) + 0.3(0.4)^2 + \cdots + 0.3(0.4)^{n-1}.$$

Thus (C_n) is the sequence of partial sums for the geometric series

$$\sum_{n=1}^{\infty} 0.3(0.4)^{n-1}.$$

We have a formula for the value of C_n, namely

$$C_n = \frac{a(1-r^n)}{1-r} = \frac{0.3(1-(0.4)^n)}{1-0.4} = \frac{1-(0.4)^n}{2}.$$

The sequence (C_n) converges to

$$\lim_{n\to\infty} C_n = \frac{1}{2}.$$