Math 113 Lecture #27
§11.3: The Integral Test and Estimates of Sums

The Integral Test. So far, we have had to compute the limit of the sequence of the partial sums to determine if a series converges or diverges:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n.$$

We now develop a powerful tool for detecting convergence or divergence of series whose terms are positive and decreasing, i.e., we assume

$$a_n \geq 0 \text{ and } a_{n+1} < a_n \text{ for all } n.$$

Theorem. Suppose there is a continuous, positive, decreasing function f defined on $[1, \infty)$ such that $a_i = f(i)$. The series

$$\sum_{i=1}^{\infty} a_i$$

converges if and only if the improper integral

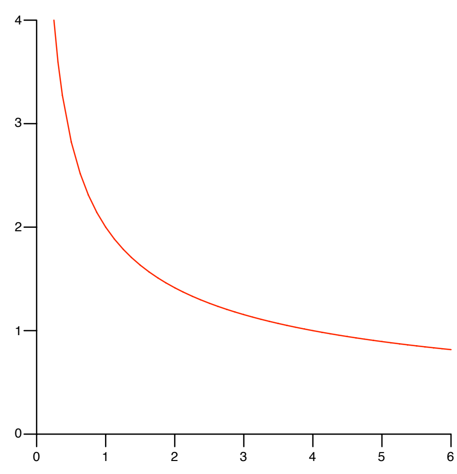
$$\int_1^{\infty} f(t)\, dt$$

converges.

Remark. Since this is an "if and only if" theorem, we can replace both occurrences of *converges* with *diverges* and it remains valid.

Proof. Suppose that the improper integral is convergent.

We will compare the area under the graph of f with the terms in the series. Here is the graph of a typical f.



The rectangle with base between $i-1$ and i and height $f(i)$ has area $f(i) = a_i$.

So between $i = 1$ and $i = 2$ we have a rectangle whose area is a_2, between $i = 2$ and $i = 3$ a rectangle whose area is a_3, etc.

The addition of the area of these rectangles up to $i = n$ is almost the n^{th}-partial sum:

$$a_2 + a_3 + \cdots + a_n = s_n - a_1.$$

Since each of the rectangles lies beneath the graph of f, we have

$$s_n - a_1 \leq \int_1^n f(t)\ dt.$$

Since f is a positive function, and the improper integral of f converges, we have

$$s_n - a_1 \leq \int_1^n f(t)\ dt \leq \int_1^\infty f(t)\ dt.$$

Solving the inequality for s_n gives

$$s_n \leq a_1 + \int_1^\infty f(t)\ dt.$$

The right hand side of this inequality is a constant that does not depend on n, and hence is an upper bound for the sequence of partial sums.

Since $a_{n+1} = f(n+1) > 0$, the sequence of partial sums is increasing:

$$s_{n+1} = s_n + a_{n+1} > s_n.$$

We apply the Monotonic Sequence Theorem to the partial sums to show that $\{s_n\}$ is a convergent sequence.

Thus, the series is convergent.

Now suppose that the improper integral of f over $[1, \infty)$ is divergent.

We form a rectangle with base between $i-1$ and i with height $f(i-1)$ whose area is $f(i-1) = a_{i-1}$.

Since f is decreasing, the tops of these rectangles lie above the graph of f.

The sum of the area of these rectangles up to $i = n$ is bounded below by the area under the graph of f over $[1, n]$:

$$\int_1^n f(t)\ dt \leq a_1 + a_2 + \cdots a_{n-1} = s_{n-1}.$$

Since f is positive, the only way for the improper integral of f over $[1, \infty)$ to diverge is for it to go to infinity:

$$\infty = \int_1^\infty f(t)\ dt = \lim_{n\to\infty} \int_1^n f(t)\ dt \leq \lim_{n\to\infty} s_{n-1}.$$

This means that the sequence of partial sums diverges, and so the series diverges. $\square$.

Example 1. For what values of p do we have convergence for the *p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}?$$

For $p = 1$ this is the Harmonic series, which we will show diverges by the Integral Test.

Since $a_n = f(n)$ for $f(t) = 1/t^p$, a continuous, positive, decreasing function, we apply the integral test to check for convergence or divergence:

$$\int_1^{\infty} f(t)\ dt = \lim_{A\to\infty} \int_1^A \frac{1}{t^p}\ dt = \lim_{A\to\infty} \begin{cases} \ln A - \ln 1 & \text{if } p = 1, \\ \dfrac{1}{(1-p)A^{p-1}} - \dfrac{1}{1-p} & \text{if } p \neq 1. \end{cases}$$

The limit does not exist when $p = 1$ or $p < 1$, but does exist when $p > 1$.

So the p-series converges when $p > 1$ and diverges for $p \leq 1$.

Estimate of Sums.

For a convergence series, anyone of the partial sums is an approximation to the value of the series since

$$s_n \to s = \sum_{i=1}^{\infty} a_i.$$

The **remainder** determines how good of an approximation any particular partial sum is:

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$$

In other words, the sum of the "tail" of the terms in the series determines R_n.

Assuming that $a_n = f(n)$ for a continuous, positive, decreasing function f defined on $[1, \infty)$, we use rectangles like we constructed in the proof of the Integral Test to give the following two inequalities that the remainder satisfies:

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} \cdots \leq \int_n^{\infty} f(t)\ dt,$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} \cdots \geq \int_{n+1}^{\infty} f(t)\ dt.$$

Combining these two inequalities gives the **Remainder Estimate for the Integral Test**:

$$\int_{n+1}^{\infty} f(t)\ dt \leq R_n \leq \int_n^{\infty} f(t)\ dt.$$

We can substitute $s - s_n$ in these inequalities for R_n, and solving for s in the middle by itself gives

$$s_n + \int_{n+1}^{\infty} f(t)\ dt \leq \sum_{i=1}^{\infty} a_i \leq s_n + \int_n^{\infty} f(t)\ dt.$$

This second set of inequalities gives a better estimate of the value of the convergent series.

Example 2. We show how to use the remainder estimate for the integral test to find an approximation as close as desired to the value of the convergent p-series

$$\sum_{i=1}^{\infty} \frac{1}{n^3}.$$

For this we need the value of two improper integrals:

$$\int_{n+1}^{\infty} \frac{1}{t^3}\, dt = \lim_{A\to\infty} \left[\frac{-1}{2t^2}\right]_{n+1}^{A} = \frac{1}{2(n+1)^2},$$
$$\int_{n}^{\infty} \frac{1}{t^3}\, dt = \lim_{A\to\infty} \left[\frac{-1}{2t^2}\right]_{n}^{A} = \frac{1}{2n^2}.$$

An estimate on the remainder for the partial sum $s_{10} \approx 1.197532$ is

$$s - s_{10} = R_{10} \le \int_{10}^{\infty} \frac{1}{t^3}\, dt = \frac{1}{2(10)^2} = \frac{1}{200} = 0.005.$$

To get an a reminder that is smaller than 0.0005 requires that n satisfy

$$R_n \le \int_{n}^{\infty} \frac{1}{t^3}\, dt = \frac{1}{2n^2} = 0.0005,$$

that is, that

$$n \ge 32.$$

Rather than computing s_{32}, we apply the second set of inequalities derived above:

$$s_{10} + \frac{1}{2(11)^2} = s_{10} + \int_{10+1}^{\infty} \frac{1}{t^3}\, dt \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{t^3}\, dt = s_{10} + \frac{1}{200}.$$

This gives

$$1.201664 \le s \le 1.202532.$$

The midpoint of this interval in which s lies is at most half the length of the interval away from the actual value of s.

Thus

$$s \approx \frac{1.201664 + 1.202532}{2} = 1.2021$$

to within an error of

$$\frac{1.202532 - 1.201664}{2} = 0.000434.$$