Math 113 Lecture #28
§11.4: Comparison Tests

The Comparison Test. Oftentimes we can compare one series with another for which the convergence or divergence of one is "easy" to determine, and from this infer the convergence or divergence of the other.

This idea of comparison is a tool in addition to the Integral Test for convergence or divergence.

Theorem. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is divergent.

Proof. Suppose that $\sum b_n$ is convergent and $a_n \le b_n$ for all n, and set

$$s_n = \sum_{i=1}^{n} a_i, \quad t_n = \sum_{i=1}^{n} b_i, \text{ and } t = \sum_{i=1}^{\infty} b_i.$$

Positivity of the terms in both series implies that $\{s_n\}$ and $\{t_n\}$ are increasing sequences.

The inequalities $a_n \le b_n$ implies that $s_n \le t_n$ for all n.

Since $t_n \to t$ as $n \to \infty$ (i.e., $\sum b_n$ converges) and $\{t_n\}$ is increasing, it follows that

$$s_n \le t \text{ for all } n.$$

So $\{s_n\}$ is an increasing bounded above sequence which therefore converges by the Monotonic Sequence Theorem.

Now suppose that $\sum b_n$ diverges and that $a_n \ge b_n$ for all n.

Positivity of the b_i's implies that $\sum b_n$ diverges to ∞, i.e., $t_n \to \infty$.

The inequalities $a_n \ge b_n$ now imply that $s_n \ge t_n$ for all n.

The divergence of t_n to ∞ therefore gives the divergence of s_n to ∞ as $n \to \infty$.

The series $\sum a_n$ is divergent. $\square$

When applying the Comparison Test, we use series whose convergence or divergence is known, such as the p-series and the geometric series.

Example 1. Determine whether or not the following series converges:

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}.$$

We seek a series to compare with this, and we typically find this series by applying inequalities to the terms of the series.

We use $n - 1 < n$ for all n to do this:

$$\frac{n-1}{n^2\sqrt{n}} \le \frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

So we are comparing the original series with a p-series for $p = 3/2$.

Since $p = 3/2$ is bigger than 1,

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

converges, so that by the Comparison Test, so does the original series.

Example 2. Determine whether or not the following series converges:

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}.$$

Again we apply inequalities to the terms to get a comparison series: since $n - 1 < n$ for all $n \geq 2$, then $(n-1)^{-1} > n^{-1}$ for all $n \geq 2$, so that

$$\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}.$$

The p-series with $p = 1/2$ diverges, and so by the Comparison Test, the original series diverges too.

The Limit Comparison Test.

Here is another useful comparison type test that ties the convergence or divergence of two series together using ratios.

Theorem. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is a positive number, then both series converge or both series diverge.

Proof. Set

$$\infty > c = \lim_{n\to\infty} \frac{a_n}{b_n} > 0.$$

Choose positive numbers m and M such that $m < c < M$.

Because a_n/b_n converges to c, there is an N such that for all $n \geq N$, the following inequalities hold:

$$m \leq \frac{a_n}{b_n} \leq M \text{ for all } n \geq N.$$

Since $b_n > 0$ for all n, we can multiply these inequalities through by b_n to get

$$mb_n \leq a_n \leq Mb_n \text{ for all } n \geq N.$$

Now if $\sum b_n$ converges, then so does $\sum Mb_n = M \sum b_n$, whence by the Comparsion Test, the series $\sum a_n$ converges.

On the other hand, if $\sum a_n$ converges, then the Comparison Test shows that $\sum mb_n = m \sum b_n$ converges too, and with $m > 0$, the series $\sum b_n$ converges.

Now if $\sum b_n$ diverges, the so does $\sum mb_n = m \sum b_n$, and so by the Comparison Test, the series $\sum a_n$ diverges.

On the other hand, if $\sum a_n$ diverges, then by the Comparison Test, the series $\sum Mb_n = M \sum b_n$ diverges too, and since $0 < M < \infty$, so does $\sum b_n$. $\square$.

Example 3. Determine whether or not the following series converges:

$$\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1} = \sum_{n=1}^{\infty} a_n.$$

We find the comparison series by eliminating what we think is extraneous stuff from the a_n's:

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}.$$

The series $\sum b_n$ is a divergent p-series ($p = 1$), and

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\dfrac{n^2 - 5n}{n^3 + n + 1}}{\dfrac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n^3 - 5n^2}{n^3 + n + 1} \\ &= 1. \end{aligned}$$

Since $0 < c = 1 < \infty$, the Limit Comparison Test shows that $\sum a_n$ diverges.

Example 4. Determine whether or not the following series converges:

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n} = \sum_{n=1}^{\infty} a_n.$$

We use the inequality $(1 + 1/n)^2 \leq 4$ for all n to find a comparison series:

$$\sum_{n=1}^{\infty} 4e^{-n} = \sum_{n=1}^{\infty} b_n.$$

This is the geometric series

$$\sum_{n=1}^{\infty} \frac{4}{e^n} = \sum_{n=1}^{\infty} \frac{4}{e} \left(\frac{1}{e}\right)^{n-1}$$

which converges since $|1/e| < 1$.

We now compute the limit of the ratios of a_n and b_n:

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{4} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}.$$

Since $\sum b_n$ converges, the Limit Comparison Test shows that $\sum a_n$ converges.

Estimating Sums.

If we have shown a series $\sum a_n$ converges by comparison with a convergent series $\sum b_n$ where $0 \leq a_n \leq b_n$, then we can estimate the error of approximating the series $\sum a_n$ with a partial sum in terms of the remainders.

The remainder for $s = \sum a_n$ is

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

The remainder for $t = \sum b_n$ is

$$T_n = t - t_n = b_{n+1} + b_{n+2} + b_{n+3} + \cdots .$$

Since $a_n \le b_n$ we have that

$$R_n \le T_n.$$

If $\sum b_n$ is a convergent geometric series, then we can compute explicitly T_n.

If $\sum b_n$ is a convergent p-series, we can use the Remainder Estimate for the Integral Test to estimate T_n.

Example 5. The series

$$s = \sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$$

converges by comparison with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

An estimate on the remainder for this convergent p-series is given by

$$T_n \le \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}.$$

Thus we have

$$R_n \le T_n \le \frac{1}{2n^2}.$$

For $n = 100$, we have

$$R_{100} \le \frac{1}{2(100)^2} = 0.00005.$$

The partial sum

$$s_{100} = 0.6864538$$

is within 0.00005 of the actual value of s.